

On the scaling limits of weakly asymmetric bridges

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Abstract

We consider a discrete bridge from $(0, 0)$ to $(2N, 0)$ evolving according to the corner growth dynamics, where the jump rates are subject to an upward asymmetry of order $N^{-\alpha}$ with $\alpha \in (0, \infty)$. We provide a classification of the asymptotic behaviours - invariant measure, hydrodynamic limit and fluctuations - of this model according to the value of the parameter α .

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1 Introduction

The simple exclusion process is a statistical physics model that has received much attention from physicists and probabilists over the years. The purpose of the present article is to classify the scaling limits of a particular instance of this process, according to the asymmetry imposed on the jump rates.

Consider a system of N particles on the linear lattice $\{1, \dots, 2N\}$, subject to the exclusion rule that prevents any two particles from sharing a same site. Each particle, independently of the others, jumps to its left at rate p_N and to its right at rate $1 - p_N$ as long as the target site is not occupied. Additionally, we impose a “zero-flux” boundary condition to the system: a particle located at site 1, resp. at site $2N$, is not allowed to jump to its left, resp. to its right. At any given time t , let $X_i(t)$ be equal to $+1$ if the i -th site is occupied, and to -1 otherwise.

It is classical to associate to such a particle system a so-called height function, defined by

$$S(0) = 0, \quad S(k) = \sum_{i=1}^k X_i, \quad k = 1, \dots, 2N.$$

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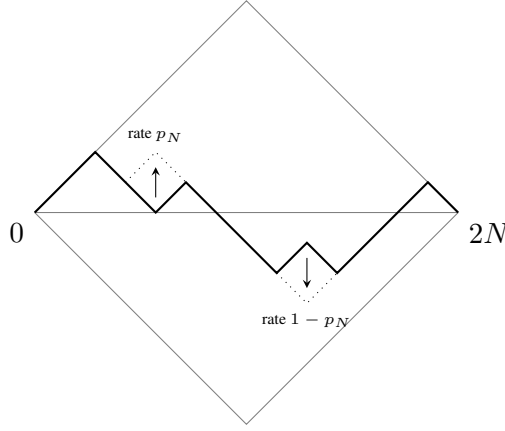


Figure 1: An example of interface.

Necessarily, $S(2N) = 0$ so that S is a discrete bridge. The dynamics of the particle system can easily be expressed at the level of the height function: at rate p_N , resp. $1 - p_N$, each downwards corner, resp. upwards corner, flips into its opposite: we refer to Figure 1 for an illustration. The law of the corresponding dynamical interface will be denoted by \mathbb{P}^N .

This dynamics admits a unique reversible probability measure:

$$\mu_N(S) = \frac{1}{Z_N} \left(\frac{p_N}{1 - p_N} \right)^{\frac{1}{2} A(S)}, \quad (1.1)$$

where $A(S) = \sum_{k=1}^{2N} S(k)$ is the area under the discrete bridge S , and Z_N is a normalisation constant, usually referred to as the partition function. This observation appears in various forms in the literature, see for instance [JL94, FS10, EL15]. Notice that the dynamics is reversible w.r.t. μ_N even if the jump rates are asymmetric: this feature of the model is a consequence of our “zero-flux” boundary condition.

From now on, we only consider “upwards” asymmetries, that is, $p_N \geq 1/2$, and we aim at understanding the behaviour of the interface according to the strength of the asymmetry. It is clear that the interface will be pushed higher and higher as the asymmetry increases. On the other hand, the interface is subject to some geometric restrictions: it is bound to 0 at both ends, and it is lower than the deterministic shape $k \mapsto k \wedge (2N - k)$. Actually, it is simple to check that under a strong asymmetry, that is, $p_N = p > 1/2$, the interface is essentially stuck to the latter deterministic shape. Therefore, to see non-trivial behaviours we need to consider asymmetries that vanish with N . We make the following choice of parametrisation:

$$\frac{p_N}{1 - p_N} = \exp \left(\frac{4\sigma}{(2N)^\alpha} \right), \quad \sigma > 0, \quad \alpha \in (0, \infty),$$

so that

$$p_N = \frac{1}{2} + \frac{\sigma}{(2N)^\alpha} + \mathcal{O}\left(\frac{1}{N^{2\alpha}}\right).$$

The important parameter is α . When it equals $+\infty$, we are in the symmetric regime, while $\alpha = 0$ corresponds to a strong asymmetry. In the present paper, we investigate the whole range $\alpha \in (0, \infty)$.

The results are divided into three parts: first, we characterise the scaling limit of the invariant measure; second, the scaling limit of the fluctuations at equilibrium; and third, we investigate the scaling limit of the dynamics out of equilibrium. As we will see, the model displays a large variety of limiting behaviours, most of them already appear in related contexts of the literature. This work can therefore be seen as a survey paper, even though the list of references is certainly not exhaustive. Let us mention that two sections have been taken from [Lab16], and have been enriched with more details and comments.

From now on, we extend S into a piecewise affine map from $[0, 2N]$ into \mathbf{R} : namely, S is affine on every interval $[k, k + 1]$. We also let L be the log-Laplace functional associated to the Bernoulli ± 1 distribution with parameter $1/2$, namely $L(h) = \log \cosh h$.

1.1 The invariant measure

The main result of this section is a Central Limit Theorem for the interface under μ_N . To state this result, we need to rescale appropriately the interface according to the strength of the asymmetry. For $\alpha \geq 1$, the space variable will be rescaled by $2N$ so that the rescaled space variable will live in $I_\alpha = [0, 1]$. On the other hand, for $\alpha < 1$, we will zoom in a window of order $(2N)^\alpha$ around the center of the lattice, hence the rescaled space variable will live in $I_\alpha^N = [-N/(2N)^\alpha, N/(2N)^\alpha]$ for any $N \geq 1$, and $I_\alpha = \mathbf{R}$ in the limit $N \rightarrow \infty$.

This being given, we introduce the curve Σ_α^N around which the fluctuations occur. One would have expected this curve to be defined as the mean of S under μ_N , but it is actually more convenient to opt for a different definition. However, Σ_α^N coincides with the mean under μ_N up to some negligible terms, see Remark 2.5 below. For all $k \in \{0, \dots, 2N\}$, we set $x_k = k/2N$ if $\alpha \geq 1$, $x_k = (k - N)/(2N)^\alpha$ if $\alpha < 1$, and

$$\Sigma_\alpha^N(x_k) = \sum_{i=1}^k L'(h_i^N), \quad h_i^N = \frac{2\sigma}{(2N)^\alpha} \left(N - i + \frac{1}{2} \right), \quad i \in \{1, \dots, 2N\}. \quad (1.2)$$

In between these discrete values x_k 's, Σ_α^N is defined by linear interpolation. Let us mention that $\Sigma_\alpha^N(x) \sim (2N)^{2-\alpha} \sigma x(1-x)$ when $\alpha > 1$, and $\Sigma_\alpha^N(x) \sim 2N \int_0^x L'(\sigma(1-2y)) dy$ when $\alpha = 1$. On the other hand, when $\alpha < 1$, Σ_α^N differs from the maximal curve $k \mapsto k \wedge (2N - k)$ only in a window of order N^α around the center of the lattice. We refer to Figure 2 for an illustration and to Equations (2.2) and (2.3) for precise formulae.

We are now ready to introduce the rescaling for the fluctuations. For $\alpha \geq 1$, we set

$$u^N(x) := \frac{S(x2N) - \Sigma_\alpha^N(x)}{\sqrt{2N}}, \quad x \in [0, 1],$$

and for $\alpha < 1$, we set

$$u^N(x) := \frac{S(N + x(2N)^\alpha) - \Sigma_\alpha^N(x)}{(2N)^{\frac{\alpha}{2}}}, \quad x \in I_\alpha^N.$$

Theorem 1.1 *Under the invariant measure μ_N , we have $u^N \xrightarrow{(d)} B_\alpha$ as $N \rightarrow \infty$. The process B_α is a Brownian bridge on $[0, 1]$ when $\alpha \in (1, \infty)$. For $\alpha =$*

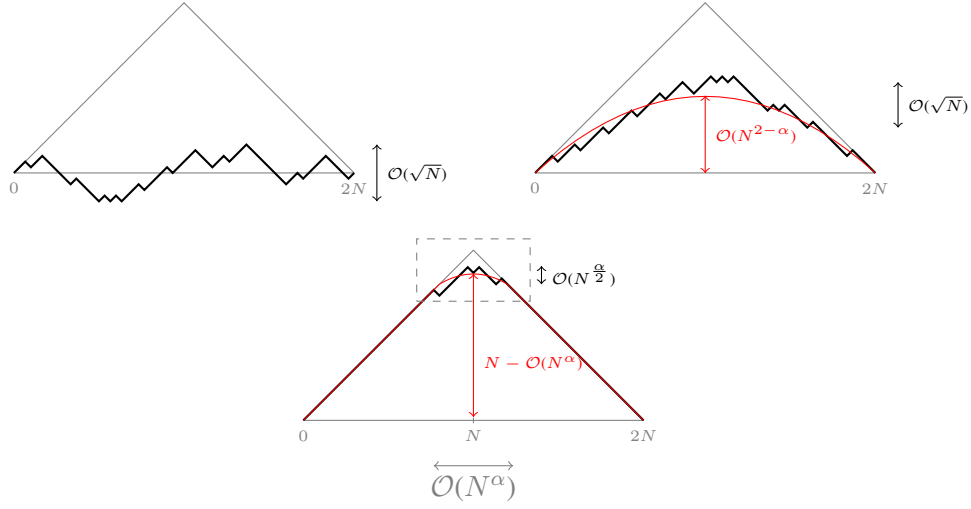


Figure 2: *Upper left $\alpha > 3/2$, upper right $\alpha \in [1, 3/2]$, bottom $\alpha < 1$. The red curve is Σ_α^N : in the first case, it is negligible compared to the fluctuations so we have not drawn it.*

1, resp. $\alpha \in (0, 1)$, it is the image of a Brownian bridge on $[0, 1]$ through a deterministic time change that maps $[0, 1]$ onto itself, resp. onto \mathbf{R} .

Remark 1.2 The covariance of B_α is given by

$$\mathbb{E}[B_\alpha(x)B_\alpha(y)] = \frac{q_\alpha(0, x)q_\alpha(y, 1)}{q_\alpha(0, 1)}, \quad \forall x \leq y \in [0, 1], \quad (1.3)$$

for $\alpha \geq 1$, and by

$$\mathbb{E}[B_\alpha(x)B_\alpha(y)] = \frac{q_\alpha(-\infty, x)q_\alpha(y, +\infty)}{q_\alpha(-\infty, +\infty)}, \quad \forall x \leq y \in \mathbf{R}, \quad (1.4)$$

for $\alpha < 1$, where

$$q_\alpha(x, y) = \begin{cases} x \vee y - x \wedge y & \text{if } \alpha \in (1, \infty) \\ \int_{x \wedge y}^{x \vee y} L''(\sigma(1 - 2u))du & \text{if } \alpha = 1 \\ \int_{x \wedge y}^{x \vee y} L''(2\sigma u)du & \text{if } \alpha \in (0, 1) \end{cases}.$$

Let us make a few comments on this result. For $\alpha > 1$, the limiting mean shape, the order of the fluctuations and the limiting law of the fluctuations are “universal”. For $\alpha = 1$, the limiting mean shape and the covariance of the limiting fluctuations depend on the log-Laplace functional of the step distribution of our static model. This is analogous with the well-known fact that the rate function of moderate deviations does not depend on the step distribution, while the rate function of large deviations does. Notice that the case $\alpha = 3/2$ is already covered in [EL15]. The case $\alpha = 1$ can be deduced from previous results of Dobrushin and Hryniv [DH96] on paths of random walks conditioned on having a given large area.

We also derive the asymptotics of the partition function Z_N .

Proposition 1.3 *As $N \rightarrow \infty$ we have*

$$\log \frac{Z_N}{2^{2N}} = \begin{cases} \frac{\sigma^2}{6}(2N)^{3-2\alpha} + \mathcal{O}(1 \vee N^{5-4\alpha}) & \text{if } \alpha \in (1, \infty), \\ (2N) \int_0^1 L(\sigma(1-2x))dx + \mathcal{O}(1) & \text{if } \alpha = 1, \\ \frac{\sigma}{2}(2N)^{2-\alpha} - 2N \log 2 + \mathcal{O}(N^\alpha) & \text{if } \alpha \in (0, 1). \end{cases}$$

Finally, let us observe that all the results presented above can be extended to a more general class of static models: namely, to paths of random walks having positive probability of coming back to 0 after $2N$ steps and whose step distribution admits exponential moments.

1.2 Fluctuations at equilibrium

We turn our attention to the dynamics. Below, \dot{W} will denote a space-time white noise on $[0, \infty) \times I_\alpha$, that is, a centred Gaussian random distribution such that for any two functions $f, g \in L^2([0, \infty) \times I_\alpha)$, we have $\mathbb{E}\dot{W}(f)\dot{W}(g) = \langle f, g \rangle$. For $\alpha \geq 1$, we set

$$u^N(t, x) := \frac{S(t(2N)^2, x2N) - \Sigma_\alpha^N(x)}{\sqrt{2N}}, \quad x \in [0, 1], t \geq 0,$$

while for $\alpha < 1$, we set

$$u^N(x) := \frac{S(t(2N)^{2\alpha}, N + x(2N)^\alpha) - \Sigma_\alpha^N(x)}{(2N)^{\frac{\alpha}{2}}}, \quad x \in I_\alpha^N, t \geq 0.$$

For convenience, we set $\Sigma_1(x) = \lim_{N \rightarrow \infty} \Sigma_1^N(x)/(2N) = \int_0^x L'(\sigma(1-2y))dy$ for all $x \in [0, 1]$, and, for $\alpha < 1$, $\Sigma_\alpha(x) = \lim_{N \rightarrow \infty} (\Sigma_\alpha^N(x) - N)/(2N)^\alpha = x + \int_{-x}^\infty (L'(2\sigma y) - 1)dy$ for all $x \in \mathbf{R}$.

Theorem 1.4 *Assume that the process starts from the invariant measure μ_N . Then, as $N \rightarrow \infty$, the process u^N converges in distribution to the process u where*

1. For $\alpha \in (1, \infty)$, u solves

$$\begin{cases} \partial_t u = \frac{1}{2} \partial_x^2 u + \dot{W}, \\ u(t, 0) = u(t, 1) = 0, \end{cases} \quad (1.5)$$

started from an independent realisation of B_α ,

2. For $\alpha = 1$, u solves

$$\begin{cases} \partial_t u = \frac{1}{2} \partial_x^2 u - 2\sigma \partial_x \Sigma_1 \partial_x u + \sqrt{1 - (\partial_x \Sigma_1)^2} \dot{W}, & x \in (0, 1) \\ u(t, 0) = u(t, 1) = 0, \end{cases} \quad (1.6)$$

started from an independent realisation of B_1 ,

3. For $\alpha \in (0, 1)$, u solves

$$\partial_t u = \frac{1}{2} \partial_x^2 u - 2\sigma \partial_x \Sigma_\alpha \partial_x u + \sqrt{1 - (\partial_x \Sigma_\alpha)^2} \dot{W}, \quad x \in \mathbf{R}, \quad (1.7)$$

started from an independent realisation of B_α .

In all cases, convergence holds in the Skorohod space $\mathbb{D}([0, \infty), \mathcal{C}(I_\alpha))$.

Once again, notice the specific behaviour when $\alpha \leq 1$: this SPDE already appears in the works of De Masi, Presutti and Scacciatelli [DMPS89] and of Dittrich and Gärtner [DG91]. Let us also cite the work of Derrida and coauthors [DELO05] on a related model interacting with reservoirs.

An important ingredient in the proof of this theorem is the Boltzmann-Gibbs principle, which is adapted to the present setting in Proposition 3.4.

1.3 Hydrodynamic limit

The subsequent question we address concerns the convergence to equilibrium: suppose we start from some initial profile S_0 at time 0, how does the interface reach its stationary state ? We consider the following rescaled height function

$$m^N(t, x) := \frac{S(t(2N)^{(\alpha+1)\wedge 2}, x2N)}{2N}, \quad t \geq 0, \quad x \in [0, 1].$$

Notice that under this scaling, at any time $t \geq 0$ the profile $x \mapsto m^N(t, x)$ is 1-Lipschitz.

Theorem 1.5 *Let $\alpha \in (0, \infty)$. We assume that the initial profile $m^N(0, \cdot)$ is deterministic and converges uniformly to some continuous profile $m_0(\cdot)$. Then, the process m^N converges in probability, in the Skorohod space $\mathbb{D}([0, \infty), \mathcal{C}([0, 1]))$, to the deterministic process m where:*

1. If $\alpha \in (1, \infty)$, m is the unique solution of the linear heat equation

$$\begin{cases} \partial_t m = \frac{1}{2} \partial_x^2 m, \\ m(t, 0) = m(t, 1) = 0, \quad m(0, \cdot) = m_0(\cdot). \end{cases} \quad (1.8)$$

2. If $\alpha = 1$, m is the solution of the following non-linear heat equation

$$\begin{cases} \partial_t m = \frac{1}{2} \partial_x^2 m + \sigma(1 - (\partial_x m)^2), \\ m(t, 0) = m(t, 1) = 0, \quad m(0, \cdot) = m_0(\cdot). \end{cases} \quad (1.9)$$

3. If $\alpha \in (0, 1)$, m is the solution of the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t m = \sigma(1 - (\partial_x m)^2), \\ m(t, 0) = m(t, 1) = 0, \quad m(0, \cdot) = m_0(\cdot). \end{cases} \quad (1.10)$$

Compare (1.8), (1.9) and (1.10) and observe that, as α decreases, the asymmetric term becomes predominant. Notice that (1.8) and (1.9) are well-posed parabolic PDEs, while (1.10) does not admit unique weak solutions so that one needs to specify the notion of solutions considered, see below. The convergence result in the case $\alpha = 1$ is similar to the results of Kipnis, Olla and Varadhan [KOV89] and of Gärtner [Gär88] who consider the WASEP respectively on the torus and on the line \mathbf{Z} ; let us also cite the work of Enaud and Derrida [ED04] on a similar model interacting with reservoirs. The case $\alpha \in (0, 1)$ is taken from [Lab16].

Let us now be more precise on the notion of solution of (1.10) that we consider here. For any Lipschitz function m_0 , we let $\eta_0(\cdot) = (\partial_x m_0(\cdot) + 1)/2$ and we say that m is solution of (1.10) if $m(t, x) = \int_0^x (2\eta(s, y) - 1)dy$ where η is the entropy solution of the Burgers equation with zero-flux boundary condition

$$\begin{cases} \partial_t \eta = 2\sigma \partial_x (\eta(1 - \eta)) , & x \in (0, 1) , \quad t > 0 , \\ \eta(t, x)(1 - \eta(t, x)) = 0 , & x \in \{0, 1\} , \quad t > 0 , \\ \eta(0, \cdot) = \eta_0 . \end{cases} \quad (1.11)$$

The precise formulation of the associated entropy conditions is given in Proposition 4.12 and is due to Bürger, Frid and Karlsen [BFK07].

Let us mention that the first theory of solutions for this type of initial-boundary value problem was established by Bardos, Le Roux and Nédélec [BIRN79] in the BV setting with Dirichlet boundary conditions. Latter on, Otto [Ott96] extended the construction to the L^∞ setting, still with Dirichlet boundary conditions. An important feature of the solutions to this conservation law with Dirichlet boundary conditions is the way they satisfy the boundary conditions: at the boundary, the trace of the solution may not equal the boundary condition prescribed by the problem, but it satisfies instead the so-called BLN conditions. The PDE (1.11) does not impose Dirichlet boundary conditions, but zero-flux boundary condition: the solution theory was proposed by Bürger, Frid and Karlsen [BFK07] and, in that case, the solution satisfies the boundary conditions at almost every time.

It turns out that the solution of (1.11) coincides with the solution of the same PDE with appropriate Dirichlet boundary conditions: one simply needs to impose $\eta(t, 0) = 1$ and $\eta(t, 1) = 0$. Then, the proof of the theorem in that case mainly consists in showing convergence of the density of particles

$$\varrho^N(t, dx) = \frac{1}{2N} \sum_{k=1}^{2N} \eta_t^N(k) \delta_{\frac{k}{2N}}(dx) , \quad \eta_t^N(k) = \frac{X(t(2N)^{1+\alpha}, k) + 1}{2} , \quad (1.12)$$

towards the deterministic process $\varrho(t, dx) = \eta(t, x)dx$ where η is the entropy solution of the Burgers equation with the above Dirichlet conditions.

This convergence result is in the flavour of previous works of Rezakhanlou [Rez91] and Bahadoran [Bah12]. Let us mention the main differences. First, here we consider a vanishing asymmetry: therefore our time-scaling is not the usual Euler scaling as in these two references. Second, we do not impose our density of particles to start from a product measure: hence, we first prove our convergence result starting from elementary product measures, and then we use the L^1 contractivity of the solution map associated to (1.11) to extend the convergence to general initial conditions.

Observe that in the case $\alpha \in (1, 3/2)$ and under the invariant measure μ_N , the interface is of order $N^{2-\alpha} \ll N$. Therefore, when the process starts from an initial condition which is at most of order $N^{2-\alpha}$, it is natural to derive the hydrodynamic limit at this finer scale $N^{2-\alpha}$. Notice that this is no longer relevant when $\alpha \geq 3/2$ since then, the fluctuations are dominant.

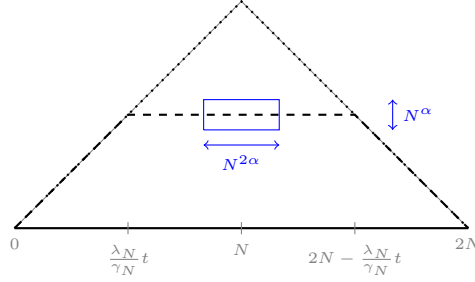


Figure 3: A plot of (1.14): the bold black line is the initial condition, the dashed line is the solution at some time $0 < t < 1/(2\sigma)$, and the dotted line is the solution at the terminal time $1/(2\sigma)$. The blue box corresponds to the window where we see KPZ fluctuations.

Theorem 1.6 For $\alpha \in (1, 3/2)$ let

$$v^N(t, x) := \frac{S(t(2N)^2, x2N)}{(2N)^{2-\alpha}}, \quad t \geq 0, \quad x \in [0, 1].$$

We assume that the initial profiles $v^N(0, \cdot)$ are deterministic, uniformly δ_i -Hölder, for some $\delta_i > 0$, and converge to some continuous profile $v_0(\cdot)$. Then, the process v^N converges in probability, in the Skorohod space $\mathbb{D}([0, \infty), \mathcal{C}([0, 1]))$, to the deterministic process v which solves the following linear heat equation

$$\begin{cases} \partial_t v = \frac{1}{2} \partial_x^2 v + \sigma, \\ v(t, 0) = v(t, 1) = 0, \quad v(0, \cdot) = v_0(\cdot). \end{cases} \quad (1.13)$$

Remark 1.7 The Hölder regularity that we impose on the initial condition does not play an important rôle: it ensures that the process is tight in a space of continuous functions from time 0.

1.4 KPZ fluctuations

From now on, we consider the flat initial condition

$$S(0, k) = k \bmod 2, \quad k \in \{0, \dots, 2N\}.$$

Let us provide explicitly the solution of the Hamilton-Jacobi equation (1.10) starting from the flat initial condition:

$$m(t, x) = x \wedge (1 - x) \wedge (\sigma t), \quad t > 0, \quad x \in [0, 1], \quad (1.14)$$

see Figure 3 for an illustration. Notice that the stationary state is reached at the finite time $T = 1/(2\sigma)$. This is an important feature of the hydrodynamic limit for $\alpha \in (0, 1)$: indeed, when $\alpha \geq 1$, the hydrodynamic limit is parabolic and reaches its stationary state in *infinite* time.

We are now interested in fluctuations around this hydrodynamic limit. The reader familiar with the Kardar Parisi Zhang (KPZ) equation would probably guess that it should arise in our setting. Let us first recall the famous result of Bertini

and Giacomin [BG97] in that direction. Consider a simple exclusion process on the infinite lattice \mathbf{Z} with jump rates $1/2 + \sqrt{\epsilon}$ to the left and $1/2$ to the right. If one starts from a flat initial profile, then results in [Gär88, DMPS89] ensure that the hydrodynamic limit grows evenly at speed $\sqrt{\epsilon}$. Then, Bertini and Giacomin look at the fluctuations around this hydrodynamic limit and show that the random process $\sqrt{\epsilon}(S(t\epsilon^{-2}, x\epsilon^{-1}) - \epsilon^{-3/2}t)$ converges to the solution of the KPZ equation, whose expression is given in (1.17) below (in Bertini and Giacomin's case, $\sigma = 1/2$).

Although our setting is similar to the one considered by Bertini and Giacomin, the “zero-flux” boundary condition induces a major difference: our process admits a reversible probability measure, while this is not the case on the infinite lattice \mathbf{Z} . However, if one starts the interface “far” from equilibrium, then we are in an irreversible setting up to the time needed by the interface to reach the stationary regime, and one would expect the fluctuations to be described by the KPZ equation.

Bertini and Giacomin's result suggests to rescale the height function by $1/(2N)^\alpha$, the space variable by $(2N)^{2\alpha}$ and the time variable by $(2N)^{4\alpha}$. The space scaling immediately forces one to take $\alpha \leq 1/2$ since, otherwise, the lattice $\{0, 1, \dots, 2N\}$ would be mapped onto a singleton in the limit. It happens that the geometry of our model imposes a further constraint: Theorem 1.5 and Equation (1.14) show that the interface reaches the stationary state in finite time in the time scale $(2N)^{\alpha+1}$; therefore, as soon as $4\alpha > \alpha + 1$, Bertini and Giacomin's scaling yields an interface which is already at equilibrium in the limit $N \rightarrow \infty$. Consequently, we have to restrict α to $(0, 1/3]$ for this scaling to be meaningful.

We set

$$h^N(t, x) := \gamma_N S(t(2N)^{4\alpha}, N + x(2N)^{2\alpha}) - \lambda_N t, \quad (1.15)$$

where

$$\gamma_N := \frac{1}{2} \log \frac{p_N}{1 - p_N}, \quad c_N := \frac{(2N)^{4\alpha}}{e^{\gamma_N} + e^{-\gamma_N}}, \quad \lambda_N := c_N(e^{\gamma_N} - 2 + e^{-\gamma_N}). \quad (1.16)$$

The following result was established in [Lab16]

Theorem 1.8 *Take $\alpha \in (0, 1/3]$ and consider the flat initial condition. As $N \rightarrow \infty$, the sequence h^N converges in distribution to the solution of the KPZ equation:*

$$\begin{cases} \partial_t h = \frac{1}{2} \partial_x^2 h - \sigma (\partial_x h)^2 + \dot{W}, & x \in \mathbf{R}, \quad t > 0, \\ h(0, x) = 0. \end{cases} \quad (1.17)$$

The convergence holds on $\mathbb{D}([0, T], \mathcal{C}(\mathbf{R}))$ where $T = 1/(2\sigma)$ when $\alpha = 1/3$, and $T = \infty$ when $\alpha < 1/3$. Here $\mathbb{D}([0, T], \mathcal{C}(\mathbf{R}))$ is endowed with the topology of uniform convergence on compact subsets of $[0, T]$.

Observe that for $\alpha = 1/3$, T is the time needed by the hydrodynamic limit to reach the stationary state. Indeed, in that case the time-scale of the hydrodynamic limit coincides with the time-scale of the KPZ fluctuations. Although one could have thought that the fluctuations continuously vanish as $t \uparrow T$, our result show that they don't: the limiting fluctuations are given by the solution of the KPZ equation, restricted to the time interval $[0, T]$. This means that the fluctuations suddenly vanish at time T ; let us give a simple explanation for this phenomenon. At any

time $t \in [0, T)$, the particle system is split into three zones: a high density zone $\{1, \dots, \frac{\lambda_N}{\gamma_N} t\}$, a low density zone $\{2N - \frac{\lambda_N}{\gamma_N} t, \dots, 2N\}$ and, in between, the bulk where the density of particles is approximately $1/2$, we refer to Figure 3. The KPZ fluctuations occur in a window of order $N^{2\alpha}$ around the middle point of the bulk: from the point of view of this window, the boundaries of the bulk are “at infinity” but move “at infinite speed”. Therefore, inside this window the system does not feel the effect of the boundary conditions until the very final time T where the boundaries of the bulk merge.

Let us recall that the KPZ equation is a singular SPDE: indeed, the solution of the linearised equation is not differentiable in space so that the non-linear term would involve the square of a distribution. While it was introduced in the physics literature [KPZ86] by Kardar, Parisi and Zhang, a first rigorous definition was given by Bertini and Giacomin [BG97] through the so-called Hopf-Cole transform $h \mapsto \xi = e^{-2\sigma h}$ that maps formally the equation (1.17) onto

$$\begin{cases} \partial_t \xi = \frac{1}{2} \partial_x^2 \xi + 2\sigma \xi \dot{W}, & x \in \mathbf{R}, \quad t > 0, \\ \xi(0, x) = 1. \end{cases} \quad (1.18)$$

This SPDE is usually referred to as the multiplicative stochastic heat equation: it admits a notion of solution via Itô integration, see for instance [DPZ92, Wal86]. Müller [Mue91] showed that the solution is strictly positive at all times, if the initial condition is non-negative and non-zero. This allows to take the logarithm of the solution, and then, one can define the solution of (1.17) to be $h := -\log \xi / 2\sigma$. This is the notion of solution that we consider in Theorem 1.8.

There exists a more direct definition of this SPDE (restricted to a bounded domain) due to Hairer [Hai13, Hai14] via his theory of regularity structures. Let us also mention the notion of “energy solution” introduced by Gonçalves and Jara [GJ14], for which uniqueness has been proved by Gubinelli and Perkowski [GP15]. It provides a new framework for characterising the solution to the KPZ equation but it requires the equation to be taken under its stationary measure.

For related convergence results towards KPZ, we refer to Amir, Corwin and Quastel [ACQ11], Dembo and Tsai [DT16], Corwin and Tsai [CT15] and Corwin, Shen and Tsai [CST16]. We also point out the reviews of Corwin [Cor12], Quastel [Qual12] and Spohn [Spo16].

The paper is organised as follows. In Section 2, we study the scaling limit of the invariant measure. Section 3 is devoted to the fluctuations at equilibrium. In Section 4 we prove Theorems 1.5 and 1.6 on the hydrodynamic limit, and in Section 5 we present the proof of the convergence of the fluctuations to the KPZ equation. Some technical bounds are postponed to the Appendix. The sections are essentially independent: at some localised places, we will rely on results obtained on the static of the model in Section 2.

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Notations. At some places in the paper, we rely on the one-to-one correspondence between particle systems and discrete interfaces. Namely, to every interface $S(k) = \sum_{i=1}^k X_i$, $1 \leq k \leq 2N$, we associate the particle system $\eta \in \{0, 1\}^{2N}$ defined by $\eta(i) = (X_i + 1)/2$ for all $i \in \{1, \dots, 2N\}$. We let τ_k denote the shift by $k \in \mathbf{Z}$, namely

$$\tau_k \eta := (\eta(k+1), \eta(k+2), \dots, \eta(k-1), \eta(k)) . \quad (1.19)$$

where indices are taken modulo $2N$.

At several occasions in the paper, we will use microscopic variables in macroscopic functions: for instance with the rescaled interface u^N with $\alpha \geq 1$, we will write $u^N(k)$ to denote $u^N(k/2N)$. This abusive notation will never raise any confusion, but will greatly simplify the notations.

2 The invariant measure

Let $\text{Be}(q)$ denote the Bernoulli ± 1 distribution with parameter $q \in [0, 1]$, and let L be the log-Laplace functional associated with $\text{Be}(1/2)$, namely $L(h) = \log \cosh h$ for all $h \in \mathbf{R}$. For each given $N \geq 1$, we will work on the set $\{-1, +1\}^{2N}$ endowed with its natural sigma-field. The canonical process will be denoted by X_1, \dots, X_{2N} , and will be viewed as the steps of the walk $S(n) := \sum_{k \leq n} X_k$. Recall that $A(S)$ is the area under the walk S , defined in (1.1).

The strategy of the proof consists in introducing an auxiliary measure ν_N which is the same as μ_N except that, under ν_N , the walk is not conditioned on coming back to 0 but satisfies $\nu_N[S(2N)] = 0$. This makes ν_N more amenable to limit theorems: we establish a Central Limit Theorem and a Local Limit Theorem for the marginals of the walk under ν_N . Since μ_N is equal to ν_N conditioned on the event $S(2N) = 0$, and since the ν_N -probability of this event can be estimated with the Local Limit Theorem, we are able to get the convergence of the marginals of the walk under μ_N . Let us now provide the details.

Let π_N be the law of the simple random walk, that is

$$\pi_N := \bigotimes_{k=1}^{2N} \text{Be}(1/2) ,$$

and let ν_N be the measure defined by

$$\frac{d\nu_N}{d\pi_N} = \frac{1}{Z'_N} \left(\frac{p_N}{1 - p_N} \right)^{\frac{A(S)}{2}} e^{\varrho_N S(2N)} , \quad Z'_N = e^{L_S(h^N)} ,$$

where $L_S(h^N) := \sum_{k=1}^{2N} L(h_k^N)$ and

$$\varrho_N = -\frac{2\sigma}{(2N)^\alpha} \left(N + \frac{1}{2} \right) , \quad h_k^N = \frac{2\sigma}{(2N)^\alpha} \left(N - k + \frac{1}{2} \right) , \quad k \in \{1, \dots, 2N\} .$$

Remark 2.1 Under the measure ν_N , the total number of particles is not equal to N almost surely, but is equal to N in mean. The measure ν_N can be seen as a mixture of $2N + 1$ measures, each of them being supported by an hyperplane of configurations with $\ell \in \{0, \dots, 2N\}$ particles. It is easy to check that our dynamics is reversible with respect to each of these measures, and therefore, with respect to ν_N .

A simple calculation yields the identities

$$\frac{d\nu_N}{d\pi_N} = \frac{1}{Z'_N} e^{\sum_k h_k^N X_k}, \quad \nu_N = \bigotimes_{k=1}^{2N} \text{Be}(q_k^N), \quad (2.1)$$

where $q_k^N = (L'(h_k^N) + 1)/2$. From there, we deduce that

$$\nu_N[S(k)] = \sum_{i=1}^k L'(h_i^N), \quad \text{Var}_{\nu_N}[S(k), S(\ell)] = \sum_{i=1}^{k \wedge \ell} L''(h_i^N).$$

Observe that the curve Σ_α^N defined in (1.2) is nothing but the mean of S under ν_N . A simple calculation then yields the following asymptotics. For $\alpha \geq 1$, we have

$$\Sigma_\alpha^N(x) = \begin{cases} (2N)^{2-\alpha} \sigma x(1-x) + \mathcal{O}(N^{4-3\alpha}) & \alpha > 1, \\ 2N \int_0^x L'(\sigma(1-2y)) dy + \mathcal{O}(1) & \alpha = 1, \end{cases} \quad (2.2)$$

and

$$\text{Var}_{\nu_N}[S(x2N), S(y2N)] = \begin{cases} (2N) q_\alpha(0, x \wedge y) + \mathcal{O}(N^{3-2\alpha}) & \alpha > 1, \\ (2N) q_\alpha(0, x \wedge y) + \mathcal{O}(1) & \alpha = 1, \end{cases}$$

for all $x, y \in [0, 1]$. For $\alpha < 1$, we find

$$\Sigma_\alpha^N(x) = N + (2N)^\alpha \left(x + \int_{-x}^\infty (L'(2\sigma y) - 1) dy \right) + \mathcal{O}(1) \quad \alpha < 1, \quad (2.3)$$

and

$$\text{Var}_{\nu_N}[S(N+x(2N)^\alpha), S(N+y(2N)^\alpha)] = (2N)^\alpha q_\alpha(-\infty, x \wedge y) + \mathcal{O}(1), \quad \alpha < 1,$$

for all $x, y \in \mathbf{R}$.

The important observation for the sequel is

$$\mu_N(\cdot) = \nu_N(\cdot \mid S(2N) = 0). \quad (2.4)$$

Indeed, if we let C_N be the set of all discrete bridges from $(0, 0)$ to $(2N, 0)$, then we have

$$Z_N = \sum_{S \in C_N} \left(\frac{p_N}{1-p_N} \right)^{\frac{1}{2}A(S)} = 2^{2N} \pi_N \left[\left(\frac{p_N}{1-p_N} \right)^{\frac{1}{2}A(S)} \mathbf{1}_{\{S(2N)=0\}}(S) \right],$$

since π_N is the uniform measure on the set (with cardinal 2^{2N}) of all lattice paths with $2N$ -steps. Hence, for any subset D of C_N , we have

$$\begin{aligned} \nu_N(D \mid S_{2N} = 0) &= \frac{\pi_N \left[\left(\frac{p_N}{1-p_N} \right)^{\frac{1}{2}A(S)} e^{-L_s(h^N)} \mathbf{1}_D(S) \right]}{\pi_N \left[\left(\frac{p_N}{1-p_N} \right)^{\frac{1}{2}A(S)} e^{-L_s(h^N)} \mathbf{1}_{\{S(2N)=0\}}(S) \right]} \\ &= \frac{1}{Z_N} 2^{2N} \pi_N \left[\left(\frac{p_N}{1-p_N} \right)^{\frac{1}{2}A(S)} \mathbf{1}_D(S) \right] \\ &= \frac{1}{Z_N} \sum_{S \in C_N} \left(\frac{p_N}{1-p_N} \right)^{\frac{1}{2}A(S)} \mathbf{1}_D(S) \end{aligned}$$

$$= \mu_N(D),$$

and (2.4) follows.

We start with a Central Limit Theorem under ν_N . Until the end of the section, k will denote an integer and $\vec{x} = (x_1, \dots, x_k)$ will be an element of $(0, 1]^k$ if $\alpha \geq 1$, of $(-\infty, +\infty]^k$ if $\alpha < 1$. It will be convenient to write $u^N(\vec{x}) = (u^N(x_1), \dots, u^N(x_k))$. In the case $\alpha < 1$, we use the convenient notation $u^N(+\infty)$ to denote $u^N(N/(2N)^\alpha)$. For all $\alpha > 0$, we also set

$$x^N := \frac{\lfloor x(2N)^{\alpha \wedge 1} \rfloor}{(2N)^{\alpha \wedge 1}}, \quad x \in \mathbf{R},$$

$x^N := +\infty$ for $x = +\infty$, and $\vec{x}^N := (x_1^N, \dots, x_k^N)$ for all \vec{x} as above.

Lemma 2.2 *The vector $u^N(\vec{x})$ under ν_N converges in distribution to $\tilde{B}_\alpha(\vec{x})$. The latter random variable is obtained as the marginals of the continuous, Gaussian process \tilde{B}_α on either $[0, 1]$ with covariance $q_\alpha(0, \cdot \wedge \cdot)$ if $\alpha \geq 1$, or on \mathbf{R} with covariance $q_\alpha(-\infty, \cdot \wedge \cdot)$ if $\alpha < 1$.*

Proof. The proof is classical, we only provide the details for the case $\alpha < 1$ as the other case is treated similarly. Until the end of the proof, i denotes the complex number $\sqrt{-1}$. For each $j \in \{1, \dots, k\}$, we define $k_j := N + \lfloor x_j(2N)^\alpha \rfloor$. For all $\vec{t} = (t_1, \dots, t_k) \in \mathbf{R}^k$, let

$$L_k(\vec{t}) := \sum_{\ell=1}^{2N} L\left(i \sum_{j=1}^k t_j \mathbf{1}_{\{\ell \leq k_j\}} + h_\ell^N\right),$$

so that $L_k(0) = L_S(h^N)$ and

$$\log \nu_N \left[e^{\sum_{j=1}^k i t_j S(N + x_j^N (2N)^\alpha)} \right] = L_k(\vec{t}) - L_k(0).$$

It is simple to check that

$$\partial_{t_j, t_m}^2 L_k|_{\vec{t}=0} = -\text{Var}_{\nu_N}[S(k_j), S(k_m)].$$

Fix $\vec{t} \in \mathbf{R}^k$. Using a Taylor expansion at the second line, we get

$$\begin{aligned} \log \nu_N \left[e^{i \langle \vec{t}, u^N(\vec{x}^N) \rangle} \right] &= L_k(\vec{t}(2N)^{-\frac{\alpha}{2}}) - L_k(0) - \frac{1}{(2N)^{\frac{\alpha}{2}}} \langle \vec{t}, \nabla L_k(0) \rangle \\ &= -\frac{1}{2} \sum_{j, \ell=1}^k t_j t_\ell q_\alpha(-\infty, x_j \wedge x_\ell) + |\vec{t}|^3 \mathcal{O}((2N)^{-\frac{\alpha}{2}}), \end{aligned}$$

so that the characteristic function of the vector $u^N(\vec{x}^N)$ converges pointwise to the characteristic function of the Gaussian vector of the statement. Since the difference between $u^N(\vec{x}^N)$ and $u^N(\vec{x})$ is negligible, the lemma follows. \square

Our next result is a Local Limit Theorem for ν_N . Let $D_\alpha^{\vec{x}, N}$ be the finite set of all $\vec{y} = (y_1, \dots, y_k) \in \mathbf{R}^k$ such that $\nu_N(u^N(\vec{x}^N) = \vec{y}) > 0$.

Lemma 2.3 *Uniformly over all $\vec{y} \in D_{\alpha}^{\vec{x}, N}$ and all $N \geq 1$, we have*

$$\frac{(2N)^{\frac{k}{2}(\alpha \wedge 1)}}{2^k} \nu_N(u^N(\vec{x}^N) = \vec{y}) - \tilde{g}_{\alpha}^{\vec{x}}(\vec{y}) = o(1),$$

where $\tilde{g}_{\alpha}^{\vec{x}}$ is the density of the random vector $\tilde{B}_{\alpha}(\vec{x})$.

Let $x_f = 1$ if $\alpha \geq 1$ and $x_f = +\infty$ if $\alpha < 1$. In the case $k = 1$, let $E_{\alpha}^{x, N}$ be the set of values y such that $\nu_N(u^N(x_f) - u^N(x^N) = y) > 0$.

Lemma 2.4 *Uniformly over all $y \in E_{\alpha}^{x, N}$ and all $N \geq 1$, we have*

$$\frac{(2N)^{\frac{1}{2}(\alpha \wedge 1)}}{2} \nu_N(u^N(x_f) - u^N(x^N) = y) - \int_{z \in \mathbf{R}} \tilde{g}_{\alpha}^{(x, x_f)}(z, z + y) dz = o(1).$$

Below, we provide the proof of the first lemma. The second lemma follows from exactly the same arguments, one simply has to notice that $u^N(x_f) - u^N(x^N)$ converges in law to $\tilde{B}_{\alpha}(x_f) - \tilde{B}_{\alpha}(x)$, and that $\int_{z \in \mathbf{R}} \tilde{g}_{\alpha}^{(x, x_f)}(z, z + y) dz$ is the density at y of this limiting r.v.

Proof of Lemma 2.3. Let us prove the case $\alpha < 1$ which is more involved. The main difference in the proof with the case $\alpha \geq 1$ lies in the fact that the forthcoming bound (2.6) cannot be applied to all h_i^N simultaneously when $\alpha < 1$. Indeed, these coefficients are not bounded uniformly over i and N when $\alpha < 1$. However, for any given $a > 0$, they are bounded uniformly over all $i \in I_{N, a} := [N - a(2N)^{\alpha}, N + a(2N)^{\alpha}]$ and all $N \geq 1$.

Without loss of generality, we can assume that $x_1 < x_2 < \dots < x_k$ so that only x_k can take the value $+\infty$. Let $\varphi_h(t) = \exp(L(h + it) - L(h))$ for $t, h \in \mathbf{R}$. This is the characteristic function of the Bernoulli ± 1 r.v. with mean $L'(h)$ so that

$$\varphi_h(t) = \cos(t) + iL'(h) \sin(t), \quad t \in \mathbf{R}, \quad h \in \mathbf{R}. \quad (2.5)$$

In particular, the characteristic function of the r.v. X_i under ν_N is given by $\varphi_{h_i^N}$. The function φ is 2π -periodic and $|\varphi_h(t)| \leq 1$ for all h, t . From (2.5), one deduces that for any compact set $K \subset \mathbf{R}$, there exists $r(K) > 0$ such that

$$|\varphi_h(t)| \leq \exp(-rt^2 L''(h)), \quad \forall t \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right], \quad \forall h \in K. \quad (2.6)$$

Let Φ_{α} denote the characteristic function of the Gaussian vector $\tilde{B}_{\alpha}(\vec{x})$. Classical arguments from Fourier analysis entail that for all $y \in D_{\alpha}^{\vec{x}, N}$

$$R_N := (2N)^{\frac{\alpha k}{2}} \nu_N(u^N(\vec{x}^N) = \vec{y}) - 2^k \tilde{g}_{\alpha}^{\vec{x}}(\vec{y}),$$

can be rewritten as

$$R_N = \frac{1}{\pi^k} \int_D \Phi_N(\vec{t}) e^{-i\langle \vec{t}, \vec{y} \rangle} d\vec{t} - \frac{1}{\pi^k} \int_{\mathbf{R}^k} \Phi_{\alpha}(\vec{t}) e^{-i\langle \vec{t}, \vec{y} \rangle} d\vec{t},$$

where Φ_N is the characteristic function of $u^N(\vec{x}^N)$ under ν_N and

$$D := \left\{ \vec{t} \in \mathbf{R}^k : |t_{\ell}| \leq \frac{\pi}{2} (2N)^{\frac{\alpha}{2}}, \ell = 1, \dots, k \right\}.$$

Notice that the factor $1/2$ in the definition of D comes from the simple fact that our step distribution charges $\{-1, 1\}$, and therefore has a maximal span equal to 2. Then, we take $\varrho \in (0, \frac{1}{2(3+k)})$ and we bound $|R_N|\pi^k$ by the sum of the following three terms

$$\begin{aligned} J_1 &= \int_{D_1} |\Phi_N(\vec{t}) - \Phi_\alpha(\vec{t})| d\vec{t}, \quad D_1 = [-N^{\varrho\alpha}, N^{\varrho\alpha}]^k, \\ J_2 &= \int_{D_2} |\Phi_\alpha(\vec{t})| d\vec{t}, \quad D_2 = \mathbf{R}^k \setminus D_1, \\ J_3 &= \int_{D_3} |\Phi_N(\vec{t})| d\vec{t}, \quad D_3 = D \setminus D_1. \end{aligned}$$

It suffices to show that these three terms vanish as $N \rightarrow \infty$. Regarding J_1 , the proof of Lemma 2.2 shows that

$$|\Phi_N(\vec{t}) - \Phi_\alpha(\vec{t})| \lesssim |\Phi_\alpha(\vec{t})| \frac{|\vec{t}|^3}{N^{\frac{\alpha}{2}}},$$

uniformly over all $|\vec{t}|^3 = o(N^{\alpha/2})$. Since $\varrho(3+k) < \frac{1}{2}$, a simple calculation shows that J_1 goes to 0 as $N \rightarrow \infty$. The convergence of J_2 to 0 as $N \rightarrow \infty$ is immediate. We turn to J_3 . For each $\ell \in \{1, \dots, k\}$, we set

$$D_{3,\ell} = D_3 \cap \left\{ |t_\ell| > 3^{-\ell} N^{\varrho\alpha}; \forall j > \ell, |t_j| \leq 3^{-j} N^{\varrho\alpha} \right\},$$

so that $D_3 = \cup_\ell D_{3,\ell}$. The important feature of these sets is that for all N large enough

$$\frac{|t_\ell|}{2(2N)^{\frac{\alpha}{2}}} \leq \frac{|t_\ell + \dots + t_k|}{(2N)^{\frac{\alpha}{2}}} \leq \frac{2\pi}{3}, \quad \forall \vec{t} \in D_{3,\ell}, \quad \forall \ell \in \{1, \dots, k\}. \quad (2.7)$$

We bound separately each term $J_{3,\ell}$ arising from the restriction of the integral in J_3 to $D_{3,\ell}$. Take $a > 0$ such that $-a < x_1 < x_{k-1} < a$ and recall that x_k can be infinite. Let K be a compact set that contains all the values h_i^N , $i \in I_{N,a}$, and let r be the corresponding constant introduced above (2.6). We also define $j_p = N + \lfloor x_p(2N)^\alpha \rfloor$ for all $p \in \{1, \dots, k\}$ and

$$I_{N,a,\ell} := I_{N,a} \cap (N + x_{\ell-1}(2N)^\alpha, N + x_\ell(2N)^\alpha].$$

Using the independence of the X_i 's under ν_N and the fact that the modulus of a characteristic function is smaller than 1 at the second line, as well as (2.6) and (2.7) at the third line, we get

$$\begin{aligned} |\Phi_N(\vec{t})| &= \left| \nu_N \left[\exp \left(i \sum_{j=1}^{2N} X(j) \sum_{p=1}^k \mathbf{1}_{\{j \leq j_p\}} \frac{t_p}{(2N)^{\frac{\alpha}{2}}} \right) \right] \right| \\ &\leq \prod_{j \in I_{N,a,\ell}} \left| \varphi_{h_j^N} \left(\frac{t_\ell + \dots + t_k}{(2N)^{\frac{\alpha}{2}}} \right) \right| \\ &\leq \exp \left(-r \frac{(t_\ell + \dots + t_k)^2}{(2N)^\alpha} \sum_{j \in I_{N,a,\ell}} L''(h_j^N) \right). \end{aligned} \quad (2.8)$$

For all N large enough we have

$$\frac{1}{(2N)^\alpha} \sum_{j \in I_{N,a,\ell}} L''(h_j^N) \geq \frac{1}{2} q(x_{\ell-1}, x_\ell \wedge a),$$

so that, using (2.7), we get

$$J_{3,\ell} \leq \int_{D_{3,\ell}} e^{-\frac{r}{8} t_\ell^2 q(x_{\ell-1}, x_\ell \wedge a)} d\vec{t} \lesssim N^{\frac{(k-1)\alpha}{2}} \int_{|t_\ell| > 3^{-\ell} N^{\alpha}} e^{-\frac{r}{8} t_\ell^2 q(x_{\ell-1}, x_\ell \wedge a)} dt_\ell,$$

which goes to 0 as $N \rightarrow \infty$. This concludes the proof. \square

Remark 2.5 It is possible to push the expansion of the local limit theorem one step further, in the spirit of [Pet75, Thm VII.12]. Then, a simple calculation shows the following. Let $x \in (0, 1)$ if $\alpha \geq 1$, and $x \in \mathbf{R}$ if $\alpha < 1$. We have

$$\mu_N[S(k)] - \nu_N[S(k)] = o(N^{\frac{1 \wedge \alpha}{2}}),$$

uniformly over all $k \leq x(2N)$ if $\alpha \geq 1$, and all $k \leq N + x(2N)^\alpha$ if $\alpha < 1$.

Corollary 2.6 For $\alpha \geq 1$, let $x \in (0, 1)$, and for $\alpha < 1$ let $x \in \mathbf{R}$. Let \mathcal{G}_x^N be the sigma-field generated by all the X_k , with $k \leq x(2N)$ when $\alpha \geq 1$, with $k \leq N + x(2N)^\alpha$ when $\alpha < 1$. Then, the Radon-Nikodym derivative of μ_N with respect to ν_N , restricted to \mathcal{G}_x^N , is bounded uniformly over all $N \geq 1$.

Proof. Let $k = \lfloor x(2N) \rfloor$ if $\alpha \geq 1$, and $k = \lfloor N + x(2N)^\alpha \rfloor$ if $\alpha < 1$. Using (2.4) at the first line and the independence of the X_i 's at the second line, we get

$$\begin{aligned} \mu_N\left(\bigcap_{i=1}^k \{S(i) = y_i\}\right) &= \frac{\nu_N\left(\bigcap_{i=1}^k \{S(i) = y_i\}; S(2N) = 0\right)}{\nu_N(S(2N) = 0)} \\ &= \nu_N\left(\bigcap_{i=1}^k \{S(i) = y_i\}\right) \frac{\nu_N(S(2N) - S(k) = -y_k)}{\nu_N(S(2N) = 0)}, \end{aligned}$$

for all $y_1, \dots, y_k \in \mathbf{R}$. By Lemmas 2.3 and 2.4, the fraction on the r.h.s. is uniformly bounded over all $y_k \in \mathbf{R}$ and all $N \geq 1$, thus yielding the statement of the corollary. \square

Lemma 2.7 Let $J = [0, 1]$ if $\alpha \geq 1$, and $J = [-A, A]$ for some arbitrary $A > 0$ if $\alpha < 1$. For any $p \geq 1$ and $\beta \in (0, 1/2)$, we have

$$\sup_{N \geq 1} \mu_N \left[\|u^N\|_{\mathcal{C}^\beta(J, \mathbf{R})}^p \right] < \infty.$$

As a consequence, the sequence of processes u^N under μ_N is tight in $\mathcal{C}([0, 1], \mathbf{R})$ if $\alpha \geq 1$, in $\mathcal{C}(\mathbf{R}, \mathbf{R})$ if $\alpha < 1$.

Proof. Observe that the law of u^N under μ_N is invariant under the reparametrisation $x \mapsto 1 - x$ when $\alpha \geq 1$, and $x \mapsto -x$ when $\alpha < 1$. Therefore, it suffices to prove the statement of the lemma with the β -Hölder norm restricted to $[0, 1/2]$ in the first case, and to $[-A, 0]$ in the second case. The uniform absolute continuity of

Corollary 2.6 ensures in turn that it suffices to bound, uniformly over all $N \geq 1$, the p -th moment of the β -Hölder norm of u^N under ν_N .

First, we bound the p -th moment of $u^N(0)$. In the case $\alpha \geq 1$, the latter is actually equal to 0, while for $\alpha < 1$, we have for any $\lambda \in \mathbf{R}$

$$\log \nu_N[e^{\lambda u^N(0)}] = \frac{\lambda^2}{2} q_\alpha(-\infty, 0) + \mathcal{O}(N^{-\alpha/2}),$$

uniformly over all $N \geq 1$, which ensures that all the moments of $u^N(0)$ are uniformly bounded in $N \geq 1$.

Regarding the Hölder semi-norm, a direct computation shows that for all $\delta > 0$

$$\log \nu_N \left[\exp \left(\frac{u^N(y) - u^N(x)}{|y - x|^\delta} \right) \right] \leq \|L''\|_\infty |x - y|^{1-2\delta},$$

uniformly over all x, y of the form $k/2N$ with $k \in \{1, \dots, 2N\}$ when $\alpha \geq 1$, and of the form $(k - N)/(2N)^\alpha$ with $k \in \{N - \lfloor A(2N)^\alpha \rfloor, \dots, N + \lfloor A(2N)^\alpha \rfloor\}$ when $\alpha < 1$. Taking $\delta \in (0, 1/2)$, this yields a finite bound uniformly over all $N \geq 1$ and all such discrete x, y . Using classical interpolation arguments, we deduce that this bound is still finite for non-discrete x, y lying in $[0, 1]$ or $[-A, A]$. Henceforth, the Kolmogorov Continuity Theorem ensures that for any $\beta \in (0, 1/2)$ and any $p \geq 1$ we have:

$$\sup_{N \geq 1} \nu_N \left[\sup_{x \neq y \in J} \frac{|u^N(x) - u^N(y)|^p}{|x - y|^{p\beta}} \right] < \infty,$$

where $J = [0, 1]$ for $\alpha \geq 1$ and $J = [-A, +A]$ for $\alpha < 1$. Notice that we did not introduce a different notation for the modification built from the Kolmogorov Continuity Theorem, since it necessarily coincides almost surely with the continuous process u^N . This concludes the proof. \square

Let B_α be the process obtained by conditioning \tilde{B}_α to be null at x_f , where $x_f = 1$ if $\alpha \geq 1$, and at $x_f = +\infty$ if $\alpha < 1$. It is a simple calculation to check that the density of the vector $(B_\alpha(x_1), \dots, B_\alpha(x_k))$ is given by

$$g_\alpha^{x_1, \dots, x_k}(y_1, \dots, y_k) = \frac{\tilde{g}_\alpha^{x_1, \dots, x_k, x_f}(y_1, \dots, y_k, 0)}{\tilde{g}_\alpha^{x_f}(0)}.$$

Proof of Theorem 1.1. By Lemma 2.7, we know that the sequence of processes u^N under μ_N is tight, so we only need to identify the limit. Let $k \geq 1$ and x_1, \dots, x_k be in $(0, 1)^k$ if $\alpha \geq 1$, and in \mathbf{R}^k if $\alpha < 1$. From (2.4), we deduce that

$$\mu_N(u^N(\vec{x}^N) = \vec{y}) = \frac{\nu_N(u^N(\vec{x}^N) = \vec{y}; u^N(x_f) = 0)}{\nu_N(u^N(x_f) = 0)}.$$

By Lemma 2.3, we have

$$\mu_N(u^N(\vec{x}^N) = \vec{y}) = 2^k (2N)^{-\frac{k}{2}(1 \wedge \alpha)} g_\alpha^{\vec{x}}(\vec{y})(1 + o(1)),$$

uniformly over all $N \geq 1$, and all \vec{y} lying in the intersection of $D_\alpha^{k, N}$ with a compact domain of \mathbf{R}^k . Thus, we deduce that for all $\vec{y}^- < \vec{y}^+ \in \mathbf{R}^k$, we have

$$\mu_N(u^N(\vec{x}^N) \in [\vec{y}^-, \vec{y}^+]) = \sum_{\vec{y} \in [\vec{y}^-, \vec{y}^+] \cap D_\alpha^{k, N}} \mu_N(u^N(\vec{x}^N) = \vec{y})$$

$$\begin{aligned}
&= \sum_{\vec{y} \in [\vec{y}^-, \vec{y}^+] \cap D_{\alpha}^{k,N}} \left(\frac{2}{(2N)^{\frac{1 \wedge \alpha}{2}}} \right)^k g_{\alpha}^{\vec{x}}(\vec{y}) (1 + o(1)) \\
&\longrightarrow \int_{\vec{y} \in [\vec{y}^-, \vec{y}^+]} g_{\alpha}^{\vec{x}}(\vec{y}) d\vec{y},
\end{aligned}$$

as $N \rightarrow \infty$. Since $|u^N(\vec{x}) - u^N(\vec{x}^N)| = \mathcal{O}(1/N^{\frac{1 \wedge \alpha}{2}})$ uniformly over all \vec{x} , we deduce that the finite dimensional marginals of u^N under μ_N converge to those of B_{α} . This concludes the proof. \square

Proof of Proposition 1.3. Recall that π_N is the uniform measure on the set of lattice paths that make $2N$ steps and start from 0. We write

$$Z_N = 2^{2N} \pi_N \left[\left(\frac{p_N}{q_N} \right)^{\frac{1}{2} A(S)} \mathbf{1}_{\{S(2N)=0\}} \right] = 2^{2N} \nu_N(S(2N)=0) e^{L_S(h^N)}. \quad (2.9)$$

Since $\nu_N(S(2N)=0) \leq 1$, it suffices to estimate the exponential term. When $\alpha > 1$, we use the fact that $L(0) = L'(0) = L^{(3)}(0) = 0$, $L''(0) = 1$ and $\|L^{(4)}\|_{\infty} < \infty$, to get

$$\begin{aligned}
L_S(h^N) &= \sum_{i=1}^{2N} L(h_i^N) = \frac{2\sigma^2}{(2N)^{2\alpha}} L''(0) \sum_{i=1}^{2N} \left(N - i + \frac{1}{2} \right)^2 + \mathcal{O}(N^{5-4\alpha}) \\
&= \frac{\sigma^2}{6} (2N)^{3-2\alpha} + \mathcal{O}(N^{(5-4\alpha) \vee (1-2\alpha)}),
\end{aligned}$$

and the asserted result follows in that case. For $\alpha = 1$, the result follows from the convergence of Riemann approximations of integrals. Finally, when $\alpha < 1$, we use the simple facts that L is even and that $L(x) - x + \log 2$ is integrable on $[0, \infty)$ to get

$$\begin{aligned}
L_S(h^N) &= 2 \sum_{i=1}^N L(h_i^N) = 2 \sum_{i=1}^N h_i^N - 2N \log 2 + 2 \sum_{i=1}^N (L(h_i^N) - h_i^N + \log 2) \\
&= \frac{\sigma}{2} (2N)^{2-\alpha} - 2N \log 2 + \mathcal{O}(N^{\alpha}),
\end{aligned}$$

thus concluding the proof. \square

3 Equilibrium fluctuations

The goal of this section is to establish Theorem 1.4. Our method of proof is classical: first, we show tightness of the sequence of processes u^N , then we identify the limit via a martingale problem. Recall that we work under the reversible measure μ_N .

3.1 Tightness

From now on, we set $J = [0, 1]$ when $\alpha \geq 1$ and $J = [-A, +A]$ for an arbitrary value $A > 0$ when $\alpha < 1$, along with

$$e_n(x) = \begin{cases} \sqrt{2} \sin(n\pi x) & \text{for } \alpha \geq 1, \\ \frac{1}{\sqrt{A}} \sin\left(\frac{n\pi}{2A}(x + A)\right) & \text{for } \alpha < 1. \end{cases}$$

This is an orthonormal basis of $L^2(J)$. For all $\beta > 0$, we define the associated Sobolev spaces

$$H^{-\beta}(J) := \left\{ f \in S'(J) : \|f\|_{H^{-\beta}}^2 := \sum_{n \geq 1} n^{-2\beta} \langle f, e_n \rangle^2 < \infty \right\}.$$

Recall that for $\alpha < 1$, the value $A > 0$ is arbitrary. In order to prove tightness of the sequence u^N in the Skorohod space $\mathbb{D}([0, \infty), \mathcal{C}([0, 1]))$ for $\alpha > 1$ and in $\mathbb{D}([0, \infty), \mathcal{C}(\mathbf{R}))$ for $\alpha < 1$, it suffices to show that the sequence of laws of $u^N(t = 0, \cdot)$ is tight in $\mathcal{C}(J)$, and that for any $T > 0$ there exists $p > 0$ such that

$$\lim_{h \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\sup_{\substack{s, t \leq T \\ |t-s| \leq h}} \|u^N(t) - u^N(s)\|_{\mathcal{C}(J)}^p \right] = 0, \quad (3.1)$$

see for instance [Bil99, Thm 13.2].

Since we start from the stationary measure, the first condition is ensured by Lemma 2.7. To check the second condition, we proceed as follows. We introduce a piecewise linear interpolation in time \bar{u}^N of our original process by setting

$$\begin{aligned} \bar{u}^N(t, \cdot) &:= (t_N + 1 - t(2N)^{2\alpha \wedge 2}) u^N\left(\frac{t_N}{(2N)^{2\alpha \wedge 2}}, \cdot\right) \\ &\quad + (t(2N)^{2\alpha \wedge 2} - t_N) u^N\left(\frac{t_N + 1}{(2N)^{2\alpha \wedge 2}}, \cdot\right), \end{aligned}$$

where $t_N := \lfloor t(2N)^{2\alpha \wedge 2} \rfloor$.

Lemma 3.1 *For all $\beta > 1/2$ and all $p \geq 1$, we have*

$$\mathbb{E}_{\mu_N}^N \left[\|\bar{u}^N(t) - \bar{u}^N(s)\|_{H^{-\beta}(J)}^p \right]^{\frac{1}{p}} \lesssim \sqrt{t-s},$$

uniformly over all $0 \leq s \leq t \leq T$ and all $N \geq 1$.

Proof. Assume that we have the bound

$$\mathbb{E}_{\mu_N}^N \left[\|u^N(t) - u^N(s)\|_{H^{-\beta}(J)}^p \right]^{\frac{1}{p}} \lesssim \sqrt{t-s} + N^{-\frac{3}{2}(1 \wedge \alpha)}, \quad (3.2)$$

uniformly over all $0 \leq s \leq t$ and all $N \geq 1$. Let $0 \leq s \leq t \leq T$. We distinguish two cases. If $t_N = s_N$ or $t = (s_N + 1)/(2N)^{2\alpha \wedge 2}$, then $t - s \leq 1/(2N)^{2\alpha \wedge 2}$ and

$$\bar{u}^N(t, \cdot) - \bar{u}^N(s, \cdot) = (t - s)(2N)^{2\alpha \wedge 2} \left(u^N\left(\frac{s_N + 1}{(2N)^{2\alpha \wedge 2}}, \cdot\right) - u^N\left(\frac{s_N}{(2N)^{2\alpha \wedge 2}}, \cdot\right) \right),$$

so that the asserted bound follows from (3.2). If $t_N \geq s_N + 1$, then we write

$$\begin{aligned} \bar{u}^N(t, \cdot) - \bar{u}^N(s, \cdot) &= u^N\left(\frac{t_N}{(2N)^{2\alpha \wedge 2}}, \cdot\right) - u^N\left(\frac{s_N + 1}{(2N)^{2\alpha \wedge 2}}, \cdot\right) \\ &\quad + \bar{u}^N(t, \cdot) - \bar{u}^N\left(\frac{t_N}{(2N)^{2\alpha \wedge 2}}, \cdot\right) \\ &\quad + \bar{u}^N\left(\frac{s_N + 1}{(2N)^{2\alpha \wedge 2}}, \cdot\right) - \bar{u}^N(s, \cdot). \end{aligned}$$

The second and third increments on the r.h.s. can be bounded using the first case above, yielding a term of order $\sqrt{t-s}$. Regarding the first increment, either $t_N = s_{N+1}$ and it vanishes, or $t_N \geq s_{N+1} + 1$ and (3.2) yields a bound of order

$$\sqrt{\frac{t_N}{(2N)^{2\alpha\wedge 2}} - \frac{s_N + 1}{(2N)^{2\alpha\wedge 2}}} + N^{-\frac{3}{2}(1\wedge\alpha)} \lesssim \sqrt{t-s} + (t-s)^{\frac{3}{4}} \lesssim \sqrt{t-s},$$

as required. To complete the proof of the lemma, it suffices to show (3.2).

For all $n \geq 1$, we let $\hat{u}(t, n) := \int_J u(t, x) e_n(x) dx$. Since $\beta > 1/2$, (3.2) is proved as soon as we show that for all $p \geq 1$

$$\mathbb{E}_{\mu_N}^N \left[|\hat{u}(t, n) - \hat{u}(s, n)|^p \right]^{\frac{1}{p}} \lesssim \sqrt{t-s} + N^{-\frac{3}{2}(1\wedge\alpha)}, \quad (3.3)$$

uniformly over all $N \geq 1$, all $n \geq 1$ and all $0 \leq s \leq t$.

Let \mathcal{L}^N be the generator of u^N . Using the reversibility of the process, we have the following identities

$$\begin{aligned} \hat{u}(t, n) - \hat{u}(s, n) &= \int_s^t \mathcal{L}^N \hat{u}(r, n) dr + M_{s,t}^N(n), \\ \hat{u}(T - (T - t), n) - \hat{u}(T - (T - s), n) &= - \int_s^t \mathcal{L}^N \hat{u}(r, n) dr + \tilde{M}_{t,s}^N(n), \end{aligned}$$

where $M_{s,t}^N(n)$, $t \geq s$ is a martingale adapted to the natural filtration of u^N , and $\tilde{M}_{t,s}^N$, $s \leq t$ is a martingale in the reversed filtration. Summing up these two identities, we deduce that it suffices to control the p -th moment of the martingales $M_{s,t}^N(n)$ and $\tilde{M}_{t,s}^N(n)$. Using the Burkholder-Davis-Gundy inequality (6.3), we get

$$\mathbb{E}_{\mu_N}^N \left[|M_{s,t}^N(n)|^p \right]^{\frac{1}{p}} \lesssim \mathbb{E}_{\mu_N}^N \left[\langle M_{s,\cdot}^N(n) \rangle_t^{p/2} \right]^{\frac{1}{p}} + \mathbb{E}_{\mu_N}^N \left[\sup_{r \in (s,t]} |M_{s,r}^N(n) - M_{s,r-}^N(n)|^p \right]^{\frac{1}{p}},$$

uniformly over all $N \geq 1$, all $n \geq 1$ and all $0 \leq s \leq t$. It is then a simple calculation to check that (3.3) is satisfied. The same bound holds for the reversed martingale by symmetry, thus concluding the proof. \square

We need an interpolation inequality to conclude the proof of the tightness.

Lemma 3.2 *Let $\eta = 1/2 - \epsilon$ and $\beta = 1/2 + \epsilon$. For $\epsilon > 0$ small enough, there exist $c > 0$ and $\gamma, \kappa \in (0, 1)$ such that*

$$\|f\|_{\mathcal{C}^\gamma(J)} \leq c \|f\|_{\mathcal{C}^\eta(J)}^\kappa \|f\|_{H^{-\beta}(J)}^{1-\kappa}, \quad \forall f \in \mathcal{C}^\eta(J) \cap H^{-\beta}(J). \quad (3.4)$$

Proof. We rely on two classical interpolation results, we refer to the book of Triebel [Tri78] for the proofs. For $q \geq 1$ and $\delta \in (0, 1)$, let $W^{\delta,q}(J)$ be the space of functions $f : J \rightarrow \mathbf{R}$ such that

$$\|f\|_{W^{\delta,q}} := \|f\|_{L^q} + \left(\int_s \int_t \frac{|f(t) - f(s)|^q}{|t-s|^{\delta q + 1}} ds dt \right)^{\frac{1}{q}} < \infty.$$

For $\eta, \beta > 0$ and $\kappa \in (0, 1)$, we set $\delta := \kappa\eta - (1 - \kappa)\beta$ as well as $q := 2/(1 - \kappa)$. Then, there exists $c' > 0$ such that

$$\|f\|_{W^{\delta,q}} \leq c' \|f\|_{\mathcal{C}^\eta}^\kappa \|f\|_{H^{-\beta}}^{1-\kappa}, \quad \forall f \in \mathcal{C}^\eta \cap H^{-\beta}.$$

Furthermore, for any $\gamma > 0$ such that $(\delta - \gamma)q > 1$ there exists $c'' > 0$ such that

$$\|f\|_{\mathcal{C}^\gamma} \leq c'' \|f\|_{W^{\delta,q}}, \quad \forall f \in W^{\delta,q}.$$

Therefore, taking $\kappa \in (2/3, 1)$, $\eta = 1/2 - \epsilon$ and $\beta = 1/2 + \epsilon$ with ϵ small enough, we deduce the statement of the lemma. \square

Using Lemma 2.7, the stationarity of the process u^N and the definition of \bar{u}^N , we deduce that for all $p \geq 1$ and all $\eta \in (0, 1/2)$

$$\sup_{N \geq 1} \sup_{0 \leq s \leq t} \mathbb{E}_{\mu_N}^N \left[\|\bar{u}^N(t) - \bar{u}^N(s)\|_{\mathcal{C}^\eta(J)}^p \right] < \infty.$$

Using Lemmas 3.1 and 3.2 together with Hölder's inequality, we deduce that there exist $\gamma, \kappa \in (0, 1)$ such that for all $p \geq 1$

$$\mathbb{E}_{\mu_N}^N \left[\|\bar{u}^N(t) - \bar{u}^N(s)\|_{\mathcal{C}^\gamma(J)}^p \right] \lesssim (t - s)^{\frac{p(1-\kappa)}{2}},$$

uniformly over all $N \geq 1$ and all $0 \leq s \leq t \leq T$. Applying Kolmogorov's Continuity Theorem, we deduce that for all $\nu \in (0, (1 - \kappa)/2)$ and all $p \geq 1$, we have

$$\sup_{N \geq 1} \mathbb{E}_{\mu_N}^N \left[\sup_{s \neq t \in [0, T]} \frac{\|\bar{u}^N(t) - \bar{u}^N(s)\|_{\mathcal{C}^\gamma(J)}^p}{|t - s|^{\nu p}} \right] < \infty.$$

We deduce that condition (3.1) is fulfilled by the process \bar{u}^N . The next lemma shows that u^N and \bar{u}^N are uniformly close on compact sets, so that (3.1) is also fulfilled by the process u^N , thus concluding the proof of tightness.

Lemma 3.3 *For all $p \geq 1$, $\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N [\sup_{t \leq T} \|u^N(t) - \bar{u}^N(t)\|_{\mathcal{C}(J)}^p] = 0$.*

Proof. For all $k \in \{0, \dots, 2N - 1\}$ and all $i \in \mathbf{N}$, we set

$$B_{i,k} := \left[\frac{i}{(2N)^2}, \frac{i+1}{(2N)^2} \right] \times \left[\frac{k}{2N}, \frac{k+1}{2N} \right], \quad \alpha \geq 1,$$

$$B_{i,k} := \left[\frac{i}{(2N)^{2\alpha}}, \frac{i+1}{(2N)^{2\alpha}} \right] \times \left[\frac{k-N}{(2N)^\alpha}, \frac{k+1-N}{(2N)^\alpha} \right], \quad \alpha < 1.$$

Suppose that for all $p \geq 1$ we have

$$\mathbb{E}_{\mu_N}^N \left[\sup_{(t,x) \in B_{i,k}} |u^N(t, x) - \bar{u}^N(t, x)|^p \right] \lesssim (2N)^{-(\alpha \wedge 1) \frac{p}{2}}, \quad (3.5)$$

uniformly over all $i \in \mathbf{N}$ and all $k \in \{0, \dots, 2N - 1\}$. Then, we deduce that

$$\mathbb{E}_{\mu_N}^N \left[\sup_{t \leq T} \|u^N(t) - \bar{u}^N(t)\|_{\mathcal{C}(J)}^p \right] \lesssim (2N)^{(\alpha \wedge 1)(3 - \frac{p}{2})},$$

uniformly over all $N \geq 1$. This yields the statement of the lemma for p large enough, and in turn, Jensen's inequality ensures that it holds for all $p \geq 1$. Therefore, we are left with the proof of (3.5). We have

$$|u^N(t, x) - \bar{u}^N(t, x)| \leq \sum_{j, \ell \in \{0, 1\}} |u^N(t, k + \ell) - u^N((i + j)(2N)^{-2(\alpha \wedge 1)}, k + \ell)|,$$

for all $(t, x) \in B_{i,k}$, all $i \in \mathbf{N}$, all $k \in \{0, \dots, 2N - 1\}$ and all $N \geq 1$. There are four terms in the sum. For each of them, the supremum over $(t, x) \in B_{i,k}$ of the corresponding increment is stochastically bounded by $2/(2N)^{(\alpha \wedge 1)/2}$ times a Poisson r.v. with mean 1. Computing the p -th moment of the latter yields (3.5). \square

3.2 The Boltzmann-Gibbs principle

The next result is the main ingredient that we need for the identification of the limit. We will work at the level of the particle system $\eta \in \{0, 1\}^{2N}$. Under the measure ν_N defined in (2.1), the $\eta(k)$'s are independent Bernoulli r.v. with parameter q_k^N . Let Ψ be a cylinder function, that is, there exists $r \in \mathbf{N}$ such that $\Psi : \{0, 1\}^r \rightarrow \mathbf{R}$. As soon as $r \leq 2N$, we can define $\Psi(\eta) = \Psi(\eta(1), \dots, \eta(r))$. Then, we set

$$V_\Psi^N(\eta) := \Psi(\eta) - \tilde{\Psi}_N - r\tilde{\Psi}'_N(\eta(1) - q_1^N), \quad \tilde{\Psi}_N := \nu_N[\Psi], \quad \tilde{\Psi}'_N := \partial_{q_1^N} \nu_N[\Psi].$$

as well as its shift by k denoted by τ_k where

$$\tau_k V_\Psi^N(\eta) := \Psi(\tau_k \eta) - \tau_k \tilde{\Psi}_N - r(\tau_k \tilde{\Psi}'_N)(\eta(k+1) - q_{k+1}^N),$$

where $\tau_k \tilde{\Psi}_N := \nu_N[\Psi(\tau_k \cdot)]$ and $\tau_k \tilde{\Psi}'_N := \partial_{q_{k+1}^N} \nu_N[\Psi(\tau_k \cdot)]$. Notice that V_Ψ^N and all its shifts have null expectation under ν_N .

Proposition 3.4 (Boltzmann-Gibbs principle) *Let φ be a continuous function on $[0, 1]$ if $\alpha \geq 1$, compactly supported on \mathbf{R} if $\alpha < 1$. Then for every $t > 0$ we have*

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left(\int_0^t \frac{1}{\sqrt{2N}} \sum_{k=1}^{2N} \tau_k V_\Psi^N(\eta_s) \varphi\left(\frac{k}{2N}\right) \right)^2 \right] &= 0, \quad \alpha \geq 1, \\ \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left(\int_0^t \frac{1}{(2N)^{\frac{\alpha}{2}}} \sum_{k=1}^{2N} \tau_k V_\Psi^N(\eta_s) \varphi\left(\frac{k-N}{2N}\right) \right)^2 \right] &= 0, \quad \alpha < 1. \end{aligned} \quad (3.6)$$

This type of result is classical in the literature on fluctuations of particle systems. However, our setting presents some specificities. First, our stationary measure is not a product measure, but it can be obtained by conditioning the product measure ν_N on the hyperplane of all configurations with N particles, as we did in Section 2. Second, ν_N is the product of independent but non-identically distributed Bernoulli measures; however the means of these Bernoulli measures vary “smoothly” in space. Given these differences with the usual setting, we provide the details of the proof, following the structure of the classical proof provided in [KL99, Thm 11.1.1]. We restrict ourselves to proving the case $\alpha < 1$, as the case $\alpha \geq 1$ is actually simpler.

Proof. Let $A > 0$ be such that $\text{supp } \varphi \subset [-A, A]$. We adopt the notation $\varphi(k)$ for $\varphi((k-N)/(2N)^\alpha)$ for simplicity. An important argument in the proof will be the uniform absolute continuity of μ_N w.r.t. ν_N , when the measures are restricted to the filtration generated by $\eta(1), \dots, \eta(N + A(2N)^\alpha)$, see Corollary 2.6. To prove the proposition, we let K be an integer and we decompose $\{N - A(2N)^\alpha, \dots, N + A(2N)^\alpha\}$ into M disjoint, consecutive boxes of size $2K + 1$ (except the last box that may be of smaller size), that we denote by B_i , $i = 1 \dots M$. Necessarily M is of order N^α/K . To each box B_i , we associate its interior B_i° as the subset of all points in B_i which are at distance at least $r + 1$ from the complement of B_i . This being given, we denote by $B^c = \cup_i (B_i \setminus B_i^\circ)$. We also let k_i be an arbitrary point in B_i , for each i . Then, we write

$$\frac{1}{(2N)^{\frac{\alpha}{2}}} \sum_{k=1}^{2N} \tau_k V_\Psi^N(\eta) \varphi(k) = \frac{1}{(2N)^{\frac{\alpha}{2}}} \sum_{k \in B^c} \tau_k V_\Psi^N(\eta) \varphi(k)$$

$$\begin{aligned}
& + \frac{1}{(2N)^{\frac{\alpha}{2}}} \sum_{i=1}^M \sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\eta) (\varphi(k) - \varphi(k_i)) \\
& + \frac{1}{(2N)^{\frac{\alpha}{2}}} \sum_{i=1}^M \sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\eta) \varphi(k_i) .
\end{aligned}$$

The contribution to (3.6) of the first two terms on the right can easily be proved to vanish as $N \rightarrow \infty$ and then $K \rightarrow \infty$: the main ingredients are Jensen's inequality on the time integral, the absolute continuity of μ_N w.r.t. ν_N , the independence of the $\eta(i)$'s under ν_N , the fact that the ν_N -expectation of V_Ψ^N is zero and the continuity of φ . Let us deal with the third term, which is more delicate. For each i , we set $\xi_i = (\eta(k), k \in B_i)$ and we let $L_{B_i}^N$ be the generator of our process restricted to B_i and *not* sped up by $(2N)^{2\alpha}$:

$$\begin{aligned}
L_{B_i}^N f(\xi_i) = \sum_{k, k+1 \in B_i} & (f(\xi_i^{k, k+1}) - f(\xi_i)) \left(p_N(1 - \xi_i(k)) \xi_i(k+1) \right. \\
& \left. + (1 - p_N) \xi_i(k)(1 - \xi_i(k+1)) \right) .
\end{aligned}$$

Following the calculations made at Equation (1.2) and below, in the proof of [KL99, Thm 11.1.1], we deduce that it suffices to show that

$$\begin{aligned}
\lim_{K \rightarrow \infty} \inf_f \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N & \left[\left(\int_0^t (2N)^{-\frac{\alpha}{2}} \sum_{i=1}^M \varphi(k_i) \right. \right. \\
& \left. \left. \times \left(\sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\eta_s) - L_{B_i}^N f(\xi_i(s)) \right) ds \right)^2 \right] = 0 ,
\end{aligned}$$

where the infimum is taken over all $f : \{0, 1\}^{2K+1} \rightarrow \mathbf{R}$. Using Jensen's inequality on the time integral, the stationarity of our dynamics w.r.t. μ_N and then the absolute continuity property recalled above, we bound the expectation in the last expression by a term of order

$$t^2 (2N)^{-\alpha} \nu_N \left[\left(\sum_{i=1}^M \varphi(k_i) \left(\sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\xi_i) - L_{B_i}^N f(\xi_i) \right) \right)^2 \right] ,$$

uniformly over all $N \geq 1$, all $K \geq 1$ and all $t \geq 0$. Recall that the ν_N -expectation of V_Ψ^N is zero, and observe that ν_N is reversible for our dynamics. Hence the ν_N -expectation of

$$\sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\xi_i) - L_{B_i}^N f(\xi_i)$$

is also zero. Moreover, ξ_i and ξ_j being independent under ν_N as soon as $i \neq j$, we deduce that the last bound can be rewritten as

$$t^2 (2N)^{-\alpha} \sum_{i=1}^M \varphi(k_i)^2 F_K^N(i) \lesssim \frac{t^2 \|\varphi\|}{K} \frac{1}{M} \sum_{i=1}^M F_K^N(i) ,$$

where

$$F_K^N(i) = \nu_N \left[\left(\sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\xi_i) - L_{B_i}^N f(\xi_i) \right)^2 \right] .$$

The main difference with the classical proof presented in [KL99, Thm 11.1.1] lies in the following argument. Let K and f be as above. For every $x \in (-A, A)$ there exists $j = j(N, x) \in \{1, \dots, M\}$ such that

$$|k_j - N - x(2N)^\alpha| = \min_{i \in \{1, \dots, M\}} (|k_i - N - x(2N)^\alpha|).$$

Recall the definition of q_k^N given below (2.1). As $N \rightarrow \infty$, $q_{k_j}^N$ converges to $q(x) := (1 + L'(-2\sigma x))/2$ and

$$F_K^N(j(N, x)) \rightarrow F_K(x) := \nu_K^{q(x)} \left[\left(\sum_{k=1}^{2K+1} \tau_k V_\Psi^{q(x)}(\xi) - L_K^{\text{sym}} f(\xi) \right)^2 \right],$$

where ν_K^q is the product of $2K + 1$ Bernoulli measures with parameter q , V_Ψ^q is defined by

$$V_\Psi^q(\xi) = \Psi(\xi) - \nu_K^q[\xi] - \partial_q(\nu_K^q[\xi])(\xi(1) - q),$$

and L_K^{sym} is the generator of the simple exclusion process on $\{0, 1\}^{2K+1}$, that is

$$L_K^{\text{sym}} f(\xi) = \frac{1}{2} \sum_{k=1}^{2K} (f(\xi^{k,k+1}) - f(\xi)).$$

Since $F_K^N(i)$ is bounded uniformly over all i and all $N \geq 1$, we deduce that

$$\frac{1}{M} \sum_{i=1}^M F_K^N(i) \rightarrow \frac{1}{2A} \int_{-A}^A F_K(x) dx,$$

as $N \rightarrow \infty$. Let $\mathcal{Q} = \{q(x), x \in [-A, A]\}$, and observe that it is a compact subset of $(0, 1)$. Putting everything together, we deduce that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^N \left[\left(\int_0^t (2N)^{-\frac{\alpha}{2}} \sum_{i=1}^M \varphi(k_i) \left(\sum_{k \in B_i^\circ} \tau_k V_\Psi^N(\eta_s) - L_{B_i}^N f(\xi_i(s)) \right) ds \right)^2 \right] \\ & \lesssim \frac{1}{K} \sup_{q \in \mathcal{Q}} \nu_K^q \left[\left(\sum_{k=1}^{2K+1} \tau_k V_\Psi^q(\xi) - L_K^{\text{sym}} f(\xi) \right)^2 \right], \end{aligned}$$

uniformly over all f and all K as above. The supremum on the right is achieved for some q_0 by continuity and compactness. Then, we can directly apply the arguments below Equation (1.3) in the proof of [KL99, Thm 11.1.1], which prove that the infimum over f of the latter expression vanishes as $K \rightarrow \infty$, thus concluding the proof of the Boltzmann-Gibbs principle. \square

3.3 Identification of the limit

We treat in details the convergence of the processes u^N when $\alpha < 1$, the arguments for $\alpha \geq 1$ are essentially the same. Let us introduce a few notations first. We write $\langle f, g \rangle$ for the usual $L^2(\mathbf{R}, dx)$ product as well as

$$\langle f, g \rangle_N := \frac{1}{(2N)^\alpha} \sum_{k=1}^{2N} f\left(\frac{k-N}{(2N)^\alpha}\right) g\left(\frac{k-N}{(2N)^\alpha}\right),$$

for the discrete L^2 product, and

$$\nabla f(x) := f(x + (2N)^{-\alpha}) - f(x), \quad \Delta f(x) := \nabla f(x) - \nabla f(x - (2N)^{-\alpha}),$$

for the discrete gradient and Laplacian. Let us state a classical result of the theory of stochastic PDEs.

Proposition 3.5 (Martingale problem) *Let $(u(t, x), x \in \mathbf{R}, t \geq 0)$ be a continuous process such that $\mathbb{E}[\|u(0, \cdot)\|_\infty] < \infty$ and for all $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$, the processes $M(\varphi)$ and $L(\varphi)$ are continuous martingales where*

$$\begin{aligned} M_t(\varphi) &= \langle u(t), \varphi \rangle - \langle u(0), \varphi \rangle - \frac{1}{2} \int_0^t \langle u(s), \partial_x^2 \varphi + 4\sigma \partial_x(\varphi \partial_x \Sigma_\alpha) \rangle ds, \\ L_t(\varphi) &= M_t(\varphi)^2 - t \langle \varphi, \varphi(1 - (\partial_x \Sigma_\alpha)^2) \rangle. \end{aligned}$$

Then, u solves (1.7) started from the initial profile $u(0, \cdot)$.

Proof of Theorem 1.4. We treat in details the case $\alpha < 1$. We know that $(u^N)_{N \geq 1}$ is a tight sequence in $\mathbb{D}([0, \infty), \mathcal{C}(\mathbf{R}))$. Since the sizes of the jumps are vanishing with N , any limit point lies in $\mathbb{C}([0, \infty), \mathcal{C}(\mathbf{R}))$. Let us consider an arbitrary converging subsequence $(u^{N_i})_{i \geq 1}$ and let u be its limit. We only need to check that u satisfies the Martingale problem of Proposition 3.5. Our starting point is the stochastic differential equations solved by the discrete process. Namely, for all $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$, we set

$$\begin{aligned} M_t^N(\varphi) &:= \langle u^N(t), \varphi \rangle_N - \langle u^N(0), \varphi \rangle_N - \frac{1}{2} \int_0^t \langle u^N(s), (2N)^{2\alpha} \Delta \varphi \rangle_N ds \\ &\quad - (2N)^{\frac{3\alpha}{2}} \int_0^t \left\langle \frac{1}{2} \Delta \Sigma_\alpha^N(\cdot) + (2p_N - 1) \mathbf{1}_{\{\Delta S(s, \cdot) \neq 0\}}, \varphi \right\rangle_N ds, \end{aligned}$$

where we have abbreviated $\Delta S(s(2N)^{2\alpha}, N + \cdot(2N)^\alpha)$ into $\Delta S(s, \cdot)$ for simplicity. Then, $M^N(\varphi)$ is a martingale, as well as $L_t^N(\varphi) := M_t^N(\varphi)^2 - \langle M^N(\varphi) \rangle_t$ where

$$\begin{aligned} \langle M^N(\varphi) \rangle_t &= \int_0^t 4 \langle p_N \mathbf{1}_{\{\Delta S(s, \cdot) > 0\}} + (1 - p_N) \mathbf{1}_{\{\Delta S(s, \cdot) < 0\}}, \varphi^2 \rangle_N ds \\ &= \int_0^t 2 \langle \mathbf{1}_{\{\Delta S(s, \cdot) \neq 0\}}, \varphi^2 \rangle_N ds + \mathcal{O}(N^{-\alpha}). \end{aligned}$$

Notice the similarity between $M^N(\varphi)$, $L^N(\varphi)$ and $M(\varphi)$, $L(\varphi)$. In order to pass to the limit along the subsequence N_i , we need to deal with the indicators in the expressions above. To that end, we set $\Psi(\eta) = \eta(1)(1 - \eta(2)) + (1 - \eta(1))\eta(2)$, which is nothing else than the indicator of the event $\{\Delta S(1) \neq 0\}$, and we aim at applying the Boltzmann-Gibbs principle of Proposition 3.4. A simple calculation yields

$$\tilde{\Psi}_N = q_1^N(1 - q_2^N) + (1 - q_1^N)q_2^N, \quad \tilde{\Psi}'_N = 1 - 2q_2^N, \quad 2p_N - 1 = \frac{2\sigma}{(2N)^\alpha} + \mathcal{O}(N^{-2\alpha}).$$

By Proposition 3.4, the error made upon replacing the indicators in M^N and L^N by $\tau \cdot \tilde{\Psi}_N + 2\tau \cdot \tilde{\Psi}'_N(\eta_s(\cdot + 1) - q_{\cdot+1}^N)$ vanishes in probability as $N \rightarrow \infty$. We are left with computing

$$(2N)^{\frac{\alpha}{2}} \sum_{k=1}^{2N} \left(\frac{1}{2} \Delta \Sigma_\alpha^N \left(\frac{k - N}{(2N)^\alpha} \right) + (2p_N - 1) \tau_k \tilde{\Psi}_N \right)$$

$$+ 2(2p_N - 1)\tau_k \tilde{\Psi}'_N(\eta_s(k+1) - q_{k+1}^N) \varphi\left(\frac{k-N}{(2N)^\alpha}\right),$$

as well as

$$\frac{1}{(2N)^\alpha} \sum_{k=1}^{2N} \left(\tau_k \tilde{\Psi}_N + 2\tau_k \tilde{\Psi}'_N(\eta_s(k+1) - q_{k+1}^N) \right) \varphi^2\left(\frac{k-N}{(2N)^\alpha}\right).$$

We have the identities $L'(h_k^N) = 2q_k^N - 1$ and

$$\nabla u^N\left(s, \frac{k-N}{(2N)^\alpha}\right) = \frac{2(\eta_s(k+1) - q_{k+1}^N)}{(2N)^{\alpha/2}}, \quad \tau_k \tilde{\Psi}_N = \frac{1}{2}(1 - L'(h_k^N)^2) + \mathcal{O}(N^{-\alpha}),$$

uniformly over all k . By a simple integration by parts, we get

$$\begin{aligned} & 2(2p_N - 1)(2N)^{\frac{\alpha}{2}} \sum_{k=1}^{2N} \tau_k \tilde{\Psi}'_N(\eta_s(k+1) - q_{k+1}^N) \varphi\left(\frac{k-N}{(2N)^\alpha}\right) \\ &= \frac{2\sigma}{(2N)^\alpha} \sum_{k=1}^{2N} u^N\left(s, \frac{k-N}{(2N)^\alpha}\right) (2N)^\alpha \nabla\left(L'(h_k^N) \varphi\left(\frac{k-N}{(2N)^\alpha}\right)\right) + \mathcal{O}(N^{-\alpha}). \end{aligned}$$

Furthermore, the identity $L'' = 1 - (L')^2$ together with the expression of Σ_α^N yield

$$\frac{1}{2} \Delta \Sigma_\alpha^N\left(\frac{k-N}{(2N)^\alpha}\right) + (2p_N - 1)\tau_k \tilde{\Psi}^N = \mathcal{O}(N^{-2\alpha}),$$

uniformly over all k . Putting everything together, we deduce that $M_t^N(\varphi)$ and $L_t^N(\varphi)$ converge in probability to $M_t(\varphi)$ and $L_t(\varphi)$ along the converging subsequence u^{N_i} . This completes the proof of Theorem 1.4. \square

4 Hydrodynamic limit

4.1 The replacement lemma

In this subsection, we work at the level of the particle system $\eta_t, t \geq 0$ where

$$\eta_t(k) = \frac{1 + X(t(2N)^{(1+\alpha)\wedge 2}, k)}{2}.$$

Notice that our dynamics still makes sense when the total number of particles is any integer between 0 and $2N$.

Let $r \geq 1$ be an integer and $\Phi : \{0, 1\}^r \rightarrow \mathbf{R}$. For all $\eta \in \{0, 1\}^{2N}$ and as soon as $2N \geq r$, we define

$$\Phi(\eta) := \Phi(\eta(1), \dots, \eta(r)).$$

We also introduce the expectation of Φ under a product of Bernoulli measures with parameter $a \in [0, 1]$:

$$\tilde{\Phi}(a) := \sum_{\eta \in \{0, 1\}^r} \Phi(\eta) a^{\#\{i: \eta(i)=1\}} (1-a)^{\#\{i: \eta(i)=0\}}. \quad (4.1)$$

We let $T_\ell(i) := \{i - \ell, i - \ell + 1, \dots, i + \ell\}$ be the box of size $2\ell + 1$ around site i , and for any sequence $a(k)$, $k \in \mathbf{Z}$, we define its average over $T_\ell(i)$ as follows:

$$\mathcal{M}_{T_\ell(i)} a := \frac{1}{2\ell + 1} \sum_{k=i-\ell}^{i+\ell} a(k) .$$

Recall the shift operator τ , defined in (1.19). We consider the sequence $\Phi(\eta)(k) := \Phi(\tau_k \eta)$ and the associated averages $\mathcal{M}_{T_\ell(i)} \Phi(\eta)$. In the sequel, we will need a “replacement lemma” that bounds the following quantity

$$V_\ell(\eta) = \left| \mathcal{M}_{T_\ell(0)} \Phi(\eta) - \tilde{\Phi} \left(\mathcal{M}_{T_\ell(0)} \eta \right) \right| .$$

The replacement lemma works for all initial conditions when $\alpha \geq 1$. On the other hand, when $\alpha < 1$, we make the following assumption.

Assumption 4.1 *For all $N \geq 1$, the initial condition ι_N is a product measure on $\{0, 1\}^{2N}$ of the form $\otimes_{k=1}^{2N} Be(f(k/2N))$, where $f : [0, 1] \rightarrow [0, 1]$ is assumed to be piecewise constant and does not depend on N .*

We could probably relax this assumption, but it is sufficient for our purpose.

Theorem 4.2 (Replacement lemma) *Let $\alpha \in (0, \infty)$ and let ι_N be a measure on $\{0, 1\}^{2N}$. For $\alpha \in (0, 1)$, we suppose that Assumption 4.1 is fulfilled. For every $\delta > 0$, we have*

$$\lim_{\epsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{P}_{\iota_N}^N \left(\int_0^t \frac{1}{N} \sum_{k=1}^{2N} V_{\epsilon N}(\tau_k \eta_s) ds \geq \delta \right) = 0 . \quad (4.2)$$

The proof of this theorem relies on the classical one-block and two-blocks estimates. First, let us introduce the Dirichlet form associated to our dynamics:

$$D_N(f) = - \sum_{\eta} \sqrt{f(\eta)} \mathcal{L}^N \sqrt{f(\eta)} \nu_N(\eta) ,$$

where $f : \{0, 1\}^{2N} \rightarrow \mathbf{R}_+$ and \mathcal{L}^N is the generator of our sped up process, that is

$$\begin{aligned} \mathcal{L}^N g(\eta) = (2N)^{(1+\alpha) \wedge 2} \sum_{k=1}^{2N-1} & (g(\eta^{k,k+1}) - g(\eta)) (p_N \eta(k+1)(1 - \eta(k)) \\ & + (1 - p_N) \eta(k)(1 - \eta(k+1))) , \end{aligned}$$

where $\eta^{k,k+1}$ is obtained from η by permuting the values at sites k and $k+1$.

The reference measure in the Dirichlet form is taken to be ν_N , as defined in Section 2, which is reversible for our dynamics. This is because we do not work only on the hyperplane of configurations with N particles but on the whole set $\{0, 1\}^{2N}$. In the statements of the lemmas below, the function f will always be non-negative and such that $\nu_N[f] = 1$.

Lemma 4.3 (One-block estimate) *For any $\alpha > 0$ and any $C > 0$, we have*

$$\lim_{\ell \rightarrow \infty} \overline{\lim}_{N \rightarrow \infty} \sup_{f: D_N(f) \leq C N^{(2-\alpha) \vee 1}} \frac{1}{N} \sum_{k=1}^{2N} \nu_N \left[V_\ell(\tau_k \eta) f(\eta) \right] = 0 .$$

The proof is due to Kipnis, Olla and Varadhan [KOV89].

Proof. In this proof, ξ always denotes an element of $\{0, 1\}^{2\ell+1}$ and η an element of $\{0, 1\}^{2N}$. The identity $\eta|_{T_\ell(i)} = \xi$ will be an abusive notation for $\eta(i - \ell - 1 + j) = \xi(j)$ for all $j \in \{1, \dots, 2\ell + 1\}$.

First, let us observe that we can restrict the sum over k to $R_\ell^N := \{\ell+1, \dots, 2N-\ell\}$ since the remaining terms have a negligible contributions. Second, we have the following identity

$$\frac{1}{N} \sum_{k \in R_\ell^N} \nu_N \left[V_\ell(\eta) f(\eta) \right] = \frac{1}{N} \sum_{k \in R_\ell^N} \sum_{\xi} \sum_{\eta: \eta|_{T_\ell(k)} = \xi} V_\ell(\xi) f(\eta) \nu_N(\eta) + \mathcal{O}(\ell^{-1}), \quad (4.3)$$

uniformly over all densities f . Let D^* denote the Dirichlet form of the symmetric simple exclusion process on $\{0, 1\}^{2\ell+1}$, that is

$$D^*(g) = \frac{1}{4} \sum_{\xi} 2^{-(2\ell+1)} \sum_{j=1}^{2\ell} \left(\sqrt{g(\xi^{j,j+1})} - \sqrt{g(\xi)} \right)^2.$$

Let us introduce

$$f_\ell(\xi) := \frac{1}{\#R_\ell^N} \sum_{i \in R_\ell^N} \sum_{\eta: \eta|_{T_\ell(i)} = \xi} \nu_N(\eta) f(\eta).$$

Since $\nu_N[f] = 1$, we immediately deduce that $\sum_{\xi} f_k(\xi) = 1$. Recall the inequality

$$\left(\sqrt{\sum_i a_i} - \sqrt{\sum_i b_i} \right)^2 \leq \sum_i (\sqrt{a_i} - \sqrt{b_i})^2,$$

that holds for all summable sequences $a_i, b_i \geq 0$. Using this inequality, one gets the bound

$$D^*(f_k) \lesssim \frac{\ell 2^{-2\ell}}{\#R_\ell^N (2N)^{2 \wedge (1+\alpha)}} D_N(f),$$

uniformly over all densities f , all $\ell \geq 1$ and all $N \geq 1$. Notice that $\#R_\ell^N = 2N - 2\ell - 1$. Combining the last bound with (4.3), we deduce that we only need to show

$$\lim_{\ell \rightarrow \infty} \overline{\lim}_{N \rightarrow \infty} \sup_{g: D^*(g) \leq \frac{C' \ell}{(2N)^{2(\alpha \wedge 1)}}} F(g) = 0,$$

where the supremum is taken over all $g : \{0, 1\}^{2\ell+1} \rightarrow \mathbf{R}_+$ such that $\sum_{\xi} g(\xi) = 1$, and where $F(g) := \sum_{\xi} V_\ell(\xi) g(\xi)$. By the lower semi-continuity of the Dirichlet form, $\{g : D^*(g) \leq \frac{C' \ell}{(2N)^{2(\alpha \wedge 1)}}\}$ is a closed subset of the compact set of all densities g and is therefore a compact set. Let g_N be an element for which F reaches its maximum over this compact set. We claim that

$$\lim_{N \rightarrow \infty} F(g_N) \leq \sup_{g: D^*(g)=0} F(g).$$

Indeed, there exists a subsequence of $(g_N)_N$ whose image through F converges to the l.h.s. One can extract another sub-subsequence that converges to some element

g_∞ . Necessarily g_∞ is a density and $D^*(g_\infty) = 0$, thus yielding the claim. Notice that $\{g : D^*(g) = 0\}$ is a convex combinations of the measures $\pi_{\ell,k}$, $k \in \{0, \dots, 2\ell + 1\}$ where $\pi_{\ell,k}$ is the uniform measure on the subset of $\{0, 1\}^{2\ell+1}$ with k particles (which is irreducible for our dynamics). Henceforth, we have to show

$$\lim_{\ell \rightarrow \infty} \sup_{k=0, \dots, 2\ell+1} \sum_{\xi} V_{\ell}(\xi) \pi_{\ell,k}(\xi) = 0.$$

This can be done using a Local Limit Theorem, see [KL99, Step 6 Chapter 5.4]. \square

Lemma 4.4 (Two-blocks estimate) *For any $\alpha \geq 1$ and any $C > 0$, we have*

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \sup_{f: D_N(f) \leq CN} \frac{1}{N} \sum_{k=1}^{2N-1} \frac{1}{(2\epsilon N + 1)^2} \\ & \times \nu_N \left[\sum_{j: |j-k| \leq \epsilon N} \sum_{j': |j'-k| \leq \epsilon N} \left| \mathcal{M}_{T_{\ell}(j')}(\eta) - \mathcal{M}_{T_{\ell}(j)}(\eta) \right| f(\eta) \right] = 0. \end{aligned}$$

For $\alpha < 1$, if ι_N satisfies Assumption 4.1, we have for all $t, \delta > 0$

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{2N-1} \frac{1}{(2\epsilon N + 1)^2} \sum_{j: |j-k| \leq \epsilon N} \sum_{j': |j'-k| \leq \epsilon N} \\ & \times \int_0^t \mathbb{P}_{\iota_N}^N \left(\left| \mathcal{M}_{T_{\ell}(j')}(\eta_s) - \mathcal{M}_{T_{\ell}(j)}(\eta_s) \right| \geq \delta \right) ds = 0. \end{aligned}$$

The proof in the case $\alpha \geq 1$ is due to Kipnis, Olla and Varadhan [KOV89].

Proof of Lemma 4.4, $\alpha \in [1, \infty)$. We can restrict the sum to all $k \in R_{\epsilon}^N$ where $R_{\epsilon}^N := \{\lceil \epsilon N \rceil, \dots, 2N - \lceil \epsilon N \rceil\}$, since the contribution of the remaining terms is negligible. We can also restrict the sum over j, j' to the set

$$J(k) := \{(j, j') : |j - k| \leq \epsilon N, |j' - k| \leq \epsilon N, j' > j + 2\ell\}.$$

Notice that $\#J(k) = \#J$ does not depend on k and that it is of order $(\epsilon N)^2$ as long as ℓ is small compared to ϵN . Therefore, we have to control

$$\frac{1}{N} \sum_{k \in R_{\epsilon}^N} \frac{1}{\#J} \nu_N \left[\sum_{(j, j') \in J(k)} \left| \mathcal{M}_{T_{\ell}(j')}(\eta) - \mathcal{M}_{T_{\ell}(j)}(\eta) \right| f(\eta) \right].$$

From now on, (ξ_1, ξ_2) will always denote an element of $\{0, 1\}^{2\ell+1} \times \{0, 1\}^{2\ell+1}$, and η an element of $\{0, 1\}^{2N}$. We set

$$\begin{aligned} D^1(g) &:= \frac{1}{4} \sum_{\xi_1, \xi_2} 2^{-2(2\ell+1)} \sum_{n=1}^{2\ell} \left(\sqrt{g(\xi_1^{n, n+1}, \xi_2)} - \sqrt{g(\xi_1, \xi_2)} \right)^2, \\ D^2(g) &:= \frac{1}{4} \sum_{\xi_1, \xi_2} 2^{-2(2\ell+1)} \sum_{n=1}^{2\ell} \left(\sqrt{g(\xi_1, \xi_2^{n, n+1})} - \sqrt{g(\xi_1, \xi_2)} \right)^2, \\ D^{\circ}(g) &:= \frac{1}{4} \sum_{\xi_1, \xi_2} 2^{-2(2\ell+1)} \sum_{n=1}^{2\ell} \left(\sqrt{g((\xi_1, \xi_2)^{\circ})} - \sqrt{g(\xi_1, \xi_2)} \right)^2, \end{aligned}$$

where $(\xi_1, \xi_2)^\circ$ is obtained from (ξ_1, ξ_2) upon exchanging the values of $\xi_1(\ell + 1)$ and $\xi_2(\ell + 1)$. Let us now set

$$f_\ell(\xi_1, \xi_2) := \sum_{k \in R_\epsilon^N} \frac{1}{\#R_\epsilon^N \#J} \sum_{(j, j') \in J(k)} \sum_{\substack{\eta: \\ \eta|_{T_\ell(j)} = \xi_1 \\ \eta|_{T_\ell(j')} = \xi_2}} f(\eta) \nu_N(\eta) .$$

Notice that $\sum_{\xi_1, \xi_2} f_\ell(\xi_1, \xi_2) = 1$. As in the proof of Lemma 4.3, we get the bounds

$$D^1(f_\ell) \lesssim \frac{\ell}{N} D_N(f) , \quad D^2(f_\ell) \lesssim \frac{\ell}{N} D_N(f) ,$$

uniformly over all ℓ and all densities f . On the other hand, we have the bound

$$D^\circ(f_\ell) \lesssim \sum_{\xi_1, \xi_2} \sum_{k \in R_\epsilon^N} \frac{1}{\#R_\epsilon^N \#J} \sum_{(j, j') \in J(k)} \sum_{\substack{\eta: \\ \eta|_{T_\ell(j)} = \xi_1 \\ \eta|_{T_\ell(j')} = \xi_2}} \left(\sqrt{f(\eta^{j, j'})} - \sqrt{f(\eta)} \right)^2 .$$

Observe that we have

$$\eta^{j, j'} = \left(\dots \left(\left(\dots ((\eta^{j, j+1})^{j+1, j+2}) \dots \right)^{j'-1, j'} \right)^{j'-2, j'-1} \dots \right)^{j, j+1} .$$

This induces a chain of configurations $\eta_0 = \eta, \eta_1 = \eta^{j, j+1}, \dots, \eta_{2(j'-j)-1} = \eta^{j, j'}$. Then we write

$$\left(\sqrt{f(\eta^{j, j'})} - \sqrt{f(\eta)} \right)^2 \leq (2(j' - j) - 1) \sum_{m=1}^{2(j'-j)-1} \left(\sqrt{f(\eta_m)} - \sqrt{f(\eta_{m-1})} \right)^2 .$$

A simple calculation then yields the following bound:

$$D^\circ(f_\ell) \lesssim \frac{(\epsilon N)^2}{N} \frac{D_N(f)}{(2N)^2} .$$

By similar arguments as in the proof of Lemma 4.3, we deduce that it suffices to show that

$$\overline{\lim}_{\ell \rightarrow \infty} \sup_{D^1(g) = D^2(g) = D^\circ(g) = 0} \sum_{\xi_1, \xi_2} \left| \mathcal{M}_{T_\ell(0)} \xi_1 - \mathcal{M}_{T_\ell(0)} \xi_2 \right| g(\xi_1, \xi_2) = 0 .$$

The arguments at the end of the proof of Lemma 4.3 apply again here and conclude the proof. \square

Proof of Lemma 4.4, $\alpha \in (0, 1)$. The proof is essentially due to Rezakhanlou, we only adapt the arguments in [Rez91, Lemma 6.6]. Using the coupling introduced in Subsection 4.4, we get a process (η_t^N, ζ_t^N) , $t \geq 0$, such that ζ is stationary with law $\otimes_{k=1}^{2N} \text{Be}(c)$, η_0 has law ι_N , and (η_0, ζ_0) is ordered according to its density. Lemma 4.16 shows that the number $n(t)$ of changes of sign of $k \mapsto \eta(k) - \zeta(k)$ is bounded by a constant $C > 0$ for all $t \geq 0$ and all $N \geq 1$. We deduce that on the box $T_{\epsilon N}(u)$ and for all $t \geq 0$, either $\eta_t \geq \zeta_t$ or $\eta_t \leq \zeta_t$ except for at most $C\epsilon N$ integers u 's in

$\{1, \dots, 2N\}$. Then, arguing similarly as in the proof of [Rez91, Lemma 6.6], we deduce that for all $s, \delta > 0$ we have

$$\begin{aligned} & \overline{\lim}_{\ell \rightarrow \infty} \overline{\lim}_{\epsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{2N-1} \frac{1}{(2\epsilon N + 1)^2} \sum_{j: |j-k| \leq \epsilon N} \sum_{j': |j'-k| \leq \epsilon N} \\ & \times \mathbb{P}_{\iota_N}^N \left(|\mathcal{M}_{T_\ell(j')}(\eta_s) - \mathcal{M}_{T_\ell(j)}(\eta_s)| \geq \delta \right) = 0. \end{aligned}$$

The Dominated Convergence Theorem completes the proof. \square

Proof of Theorem 4.2. Denote by P_t^N the semigroup associated to our discrete dynamics and by f_t^N the Radon-Nikodym derivative of

$$\frac{1}{t} \int_0^t \iota_N P_s^N ds,$$

with respect to ν_N . Let $G : \{0, 1\}^{2N} \rightarrow \mathbf{R}_+$. Then, for any measure ι_N we have

$$\mathbb{P}_{\iota_N}^N \left(\int_0^t G(\eta_s) ds \geq \delta \right) \leq \delta^{-1} \mathbb{E}_{\iota_N}^N \left[\int_0^t G(\eta_s) ds \right] \quad (4.4)$$

Classical arguments (see for instance Section 5.2 in [KL99]) ensure that $D_N(f_t^N)$ is bounded by $H_N(\iota_N | \nu_N) / 2t$ where H_N is the relative entropy defined by

$$H_N(\iota_N | \nu_N) := \nu_N \left[\frac{d\iota_N}{d\nu_N} \log \frac{d\iota_N}{d\nu_N} \right].$$

A simple calculation shows that, for any measure ι_N , this relative entropy is bounded by a term of order $N^{1 \vee (2-\alpha)}$. Consequently, we get

$$\mathbb{P}_{\iota_N}^N \left(\int_0^t G(\eta_s) ds \geq \delta \right) \leq \delta^{-1} t \sup_{f: D_N(f) \leq C N^{1 \vee (2-\alpha)}} \nu_N [G(\eta) f(\eta)], \quad (4.5)$$

where the supremum is taken over all $f : \{0, 1\}^{2N} \rightarrow \mathbf{R}_+$ such that $\nu_N[f] = 1$. The calculation performed on p.120 of [KOV89] yields

$$\begin{aligned} V_{\epsilon N}(\tau_k \eta) & \leq \frac{\|\tilde{\Phi}'\|_\infty}{(2\epsilon N + 1)^2} \sum_{j: |j-k| \leq \epsilon N} \sum_{j': |j'-k| \leq \epsilon N} \left| \mathcal{M}_{T_\ell(j')}(\eta) - \mathcal{M}_{T_\ell(j)}(\eta) \right| \\ & + \frac{1}{2\epsilon N + 1} \sum_{j: |j-k| \leq \epsilon N} V_\ell(\tau_j \eta) + \mathcal{O}\left(\frac{\ell}{N}\right), \end{aligned} \quad (4.6)$$

where the $\mathcal{O}(\frac{\ell}{N})$ is uniform in k and η , so that it has a negligible contribution in (4.2). Using (4.5), we bound the contribution of the second term as follows:

$$\begin{aligned} & \mathbb{P}_{\iota_N}^N \left(\int_0^t \frac{1}{N} \sum_{k=1}^{2N} \frac{1}{2\epsilon N + 1} \sum_{j: |j-k| \leq \epsilon N} V_\ell(\tau_j \eta_s) ds \geq \delta \right) \\ & \leq \delta^{-1} t \sup_{f: D_N(f) \leq C N^{1 \vee (2-\alpha)}} \nu_N \left[\frac{1}{N} \sum_{k=1}^{2N} \frac{1}{2\epsilon N + 1} \sum_{j: |j-k| \leq \epsilon N} V_\ell(\tau_j \eta) f(\eta) \right], \end{aligned}$$

so that Lemma 4.3 ensures that this term has a vanishing contribution as $N \rightarrow \infty$, $\epsilon \downarrow 0$ and then $\ell \rightarrow \infty$. Similarly, for $\alpha \geq 1$ the contribution of the first term of (4.6) is handled by Lemma 4.4 combined with (4.5) for $\alpha \geq 1$. For $\alpha < 1$, the contribution of the first term of (4.6) is dealt with as follows. Using (4.4), we get

$$\begin{aligned} & \mathbb{P}_N^N \left(\int_0^t \frac{1}{N} \sum_{k=1}^{2N} \frac{\|\tilde{\Phi}'\|_\infty}{(2\epsilon N + 1)^2} \sum_{\substack{j: |j-k| \leq \epsilon N \\ j': |j'-k| \leq \epsilon N}} \left| \mathcal{M}_{T_\ell(j')}(\eta_s) - \mathcal{M}_{T_\ell(j)}(\eta_s) \right| ds \geq \delta \right) \\ & \leq \delta^{-1} \frac{1}{N} \sum_{k=1}^{2N} \frac{\|\tilde{\Phi}'\|_\infty}{(2\epsilon N + 1)^2} \sum_{\substack{j: |j-k| \leq \epsilon N \\ j': |j'-k| \leq \epsilon N}} \int_0^t \mathbb{E}_{\iota_N}^N \left[\left| \mathcal{M}_{T_\ell(j')}(\eta_s) - \mathcal{M}_{T_\ell(j)}(\eta_s) \right| \right] ds . \end{aligned} \quad (4.7)$$

Then, for any $\kappa > 0$ we write

$$\mathbb{E}_{\iota_N}^N \left[\left| \mathcal{M}_{T_\ell(j')}(\eta_s) - \mathcal{M}_{T_\ell(j)}(\eta_s) \right| \right] \leq \mathbb{P}_{\iota_N}^N \left(\left| \mathcal{M}_{T_\ell(j')}(\eta_s) - \mathcal{M}_{T_\ell(j)}(\eta_s) \right| \geq \delta \kappa \right) + \delta \kappa ,$$

where we have used the fact that $\mathcal{M}_{T_\ell(j)}(\eta)$ belongs to $[0, 1]$ for all ℓ, j, η . By Lemma 4.4, we deduce that (4.7) goes to 0 as $N \rightarrow \infty$, $\epsilon \downarrow 0$ and $\ell \rightarrow \infty$. This concludes the proof. \square

4.2 Hydrodynamic limit: the parabolic case

The goal of this subsection is to prove Theorem 1.5 for $\alpha \in [1, \infty)$: we will write \mathbb{P}^N for the law of the process involved in these statements. Recall that we write $m^N(t, k)$ instead of $m^N(t, k/2N)$ for simplicity. To prove tightness of the sequence m^N in the Skorohod space $\mathbb{D}([0, \infty), \mathcal{C}([0, 1]))$, it suffices to show that the sequence $m^N(t = 0, \cdot)$ is tight in $\mathcal{C}([0, 1])$, and that we have for any $T > 0$

$$\lim_{h \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}^N \left[\sup_{t, s \leq T, |t-s| \leq h} \|m^N(t, \cdot) - m^N(s, \cdot)\|_\infty \right] = 0 . \quad (4.8)$$

The former is actually an hypothesis of our theorem. To prove the latter, we introduce a piecewise linear time interpolation of m^N , namely we set $t_N := \lfloor t(2N)^2 \rfloor$ and

$$\begin{aligned} \bar{m}^N(t, \cdot) &:= (t_N + 1 - t(2N)^2) m^N \left(\frac{t_N}{(2N)^2}, \cdot \right) \\ &\quad + (t(2N)^2 - t_N) m^N \left(\frac{t_N + 1}{(2N)^2}, \cdot \right) . \end{aligned}$$

First, we control the distance between m^N and \bar{m}^N .

Lemma 4.5 *For all $T > 0$, we have*

$$\lim_{N \rightarrow \infty} \mathbb{E}^N \left[\sup_{t \in [0, T]} \|m^N(t, \cdot) - \bar{m}^N(t, \cdot)\|_\infty \right] = 0 .$$

The proof of this lemma is almost the same as the proof of Lemma 3.3, so we omit it. This result ensures that it is actually sufficient to show (4.8) with m^N replaced by \bar{m}^N in order to get tightness.

The following proposition ensures that \bar{m}^N satisfies (4.8).

Proposition 4.6 *For any $T > 0$, there exists $\delta > 0$ such that*

$$\sup_{N \geq 1} \mathbb{E}^N \left[\sup_{0 \leq s < t \leq T} \frac{\|\bar{m}^N(t, \cdot) - \bar{m}^N(s, \cdot)\|_\infty}{|t - s|^\delta} \right] < \infty. \quad (4.9)$$

Before we proceed to the proof of this proposition, we need to collect a few preliminary results. The stochastic differential equations solved by the discrete process m^N are given by

$$\begin{aligned} dm^N(t, \ell) = & \frac{(2N)^2}{2} \Delta m^N(t, \ell) dt \\ & + (2N)(2p_N - 1) \mathbf{1}_{\{\Delta S(t(2N)^2, \ell) \neq 0\}} dt + dM^N(t, \ell), \end{aligned}$$

where M^N is a martingale with bracket given by

$$d\langle M^N(\cdot, \ell) \rangle_t = 4 \left(p_N \mathbf{1}_{\{\Delta S(t(2N)^2, \ell) > 0\}} + (1 - p_N) \mathbf{1}_{\{\Delta S(t(2N)^2, \ell) < 0\}} \right) dt.$$

If we let $p_t^N(k, \ell)$ be the fundamental solution of the discrete heat equation, see (6.4) with $c_N = (2N)^2/2$, then it is simple to check that we have

$$\begin{aligned} m^N(t, \ell) = & \sum_k p_t^N(k, \ell) m^N(0, k) + N_t^t(\ell) \\ & + (2N)(2p_N - 1) \int_0^t \sum_k p_{t-s}^N(k, \ell) \mathbf{1}_{\{\Delta S(s(2N)^2, k) \neq 0\}} ds, \end{aligned} \quad (4.10)$$

where $N_s^t(\ell)$ is the martingale defined by

$$N_s^t(\ell) := \int_0^s \sum_k p_{t-s}^N(k, \ell) dM^N(s, k), \quad s \in [0, t].$$

Lemma 4.7 *For all $\delta \in (0, \frac{1}{4})$, all $T > 0$ and all $p \geq 1$, we have*

$$\mathbb{E}^N \left[|m^N(t', x) - m^N(t, x)|^p \right]^{\frac{1}{p}} \lesssim |t' - t|^\delta + \frac{1}{2N},$$

uniformly over all $t', t \in [0, T]$, all $x \in [0, 1]$ and all $N \geq 1$.

Observe that the term $1/2N$ reflects the discontinuous nature of the process m^N .

Proof. Let $t' > t$. Given the expression (4.10), the increment $m^N(t', \ell) - m^N(t, \ell)$ can be written as the sum of three terms: the contribution of the initial condition, of the asymmetry and of the martingale terms. We bound separately the p -th moments of these three terms. First, we let \bar{p}^N be the fundamental solution of the discrete heat equation on the whole line \mathbf{Z} : contrary to p^N , \bar{p}^N is translation invariant. Let us also extend m^N into a function on the whole line \mathbf{Z} : we simply consider the $4N$ -periodic, odd function that coincides with m^N on $[0, 2N]$. By (6.11), we get

$$\sum_{k=1}^{2N-1} (p_{t'}^N(k, \ell) - p_t^N(k, \ell)) m^N(0, k)$$

$$\begin{aligned}
&= \sum_{k \in \mathbf{Z}} (\bar{p}_t^N(\ell - k) - \bar{p}_t^N(\ell - k)) m^N(0, k) \\
&= \sum_{k \in \mathbf{Z}} \bar{p}_t^N(k) \sum_{j \in \mathbf{Z}} \bar{p}_{t'-t}^N(j) (m^N(0, \ell - k - j) - m^N(0, \ell - k)) ,
\end{aligned}$$

so that, using the 1-Lipschitz regularity of the initial condition and Lemma 6.2, we deduce that

$$\left| \sum_{k=1}^{2N-1} (p_{t'}^N(k, \ell) - p_t^N(k, \ell)) m^N(0, k) \right| \lesssim \sum_{j \in \mathbf{Z}} \bar{p}_{t'-t}^N(j) \left(\frac{|j|}{2N} \right) \lesssim \sqrt{t' - t} ,$$

uniformly over all $\ell \in \{1, \dots, 2N - 1\}$, all $t \leq t' \in [0, T]$ and all $N \geq 1$.

We turn to the contribution of the asymmetry. Using the estimates on p^N recalled in Appendix 6.2, we get the following almost sure bound

$$\begin{aligned}
&\left| \int_0^{t'} \sum_k p_{t'-s}^N(k, \ell) \mathbf{1}_{\{\Delta S(s(2N)^2, k) \neq 0\}} ds - \int_0^t \sum_k p_{t-s}^N(k, \ell) \mathbf{1}_{\{\Delta S(s(2N)^2, k) \neq 0\}} ds \right| \\
&\leq \int_0^t \sum_k |p_{t'-s}^N(k, \ell) - p_{t-s}^N(k, \ell)| ds + \int_t^{t'} \sum_k p_{t'-s}^N(k, \ell) ds \\
&\lesssim (t' - t)^\delta ,
\end{aligned}$$

uniformly over all $\ell \in \{1, \dots, 2N - 1\}$, all $t, t' \in [0, T]$ and all $N \geq 1$. Finally, we treat the martingale term. We introduce the following martingales

$$\begin{aligned}
A_u^{t, t'}(\ell) &:= \int_t^{t+u} \sum_k p_{t'-s}^N(k, \ell) dM^N(s, k) , \quad u \in [0, t' - t] , \\
B_u^{t, t'}(\ell) &:= \int_0^u \sum_k (p_{t'-s}^N(k, \ell) - p_{t-s}^N(k, \ell)) dM^N(s, k) , \quad u \in [0, t] ,
\end{aligned}$$

and we observe that $N_{t'}^{t'}(\ell) - N_t^t(\ell) = A_{t'-t}^{t, t'}(\ell) + B_t^{t, t'}(\ell)$. We bound separately the p -th moments of these two terms. In both cases, we will apply (6.3). First, we observe that the jumps of these two martingales are almost surely bounded by a term of order $2/(2N)$. Second, by Lemma 6.1 we have the following almost sure bounds

$$\langle A_{t'-t}^{t, t'}(\ell) \rangle_{t'-t} \leq 4 \int_t^{t'} \sum_k p_{t'-s}^N(k, \ell)^2 ds \lesssim \frac{(t' - t)^{1/2}}{2N} ,$$

and

$$\langle B_t^{t, t'}(\ell) \rangle_t \leq 4 \int_0^t \sum_k (p_{t'-s}^N(k, \ell) - p_{t-s}^N(k, \ell))^2 ds \lesssim \frac{(t' - t)^{2\delta}}{2N} ,$$

uniformly over all $t < t' \in [0, T]$, all $\ell \in \{1, \dots, 2N - 1\}$ and all $N \geq 1$. Applying (6.3), we get the desired bound, thus concluding the proof. \square

Proof of Proposition 4.6. Fix $T > 0$. Recall the definition of \bar{m}^N . Arguing differently according to the relative values of $|t' - t|$ and $(2N)^{-2}$ (similarly as what we

did in the proof of Lemma 3.1), one can deduce from Lemma 4.7 that there exists $\delta \in (0, \frac{1}{4})$ such that for any $p \geq 1$

$$\mathbb{E}^N \left[|\bar{m}^N(t', x) - \bar{m}^N(t, x)|^p \right]^{\frac{1}{p}} \lesssim |t' - t|^\delta ,$$

uniformly over all $t', t \in [0, T]$, all $x \in [0, 1]$ and all $N \geq 1$. Using the 1-Lipschitz regularity in space of m^N and the definition of \bar{m}^N , we also get

$$\begin{aligned} & \mathbb{E}^N \left[|\bar{m}^N(t, x) - \bar{m}^N(t, y)|^p \right]^{\frac{1}{p}} \\ & \leq \sum_{j=0}^1 \mathbb{E}^N \left[\left| m^N \left(\frac{t_N + j}{(2N)^2}, x \right) - m^N \left(\frac{t_N + j}{(2N)^2}, y \right) \right|^p \right]^{\frac{1}{p}} \\ & \lesssim |x - y| , \end{aligned}$$

uniformly over all $x, y \in [0, 1]$, all $t \in [0, T]$ and all $N \geq 1$. Combining these two bounds, we obtain for all $p \geq 1$,

$$\mathbb{E}^N [|\bar{m}^N(t', x) - \bar{m}^N(t, y)|^p]^{\frac{1}{p}} \lesssim (|t' - t| + |x - y|)^\delta ,$$

uniformly over the same set of parameters. Kolmogorov's Continuity Theorem then ensures that \bar{m}^N admits a modification satisfying the bound stated in Proposition 4.6 uniformly in $N \geq 1$ for some $\delta > 0$. Since \bar{m}^N is already continuous, it coincides with its modification \mathbb{P}^N -a.s., thus concluding the proof. \square

We now proceed to the proof of Theorem 1.5: we argue differently in the cases $\alpha \in (1, \infty)$ and $\alpha = 1$. In both cases, we set

$$\langle f, g \rangle_N = \frac{1}{2N} \sum_{k=1}^{2N} f\left(\frac{k}{2N}\right) g\left(\frac{k}{2N}\right) .$$

Proof of Theorem 1.5, $\alpha \in (1, \infty)$. We already know that the sequence m^N , $N \geq 1$ is tight. Let m be the limit of a converging subsequence. To conclude the proof, we only need to show that for any $\varphi \in \mathcal{C}^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$, we have

$$\langle m(t), \varphi \rangle = \langle m(0), \varphi \rangle + \frac{1}{2} \int_0^t \langle m(s), \varphi'' \rangle ds . \quad (4.11)$$

This characterises the unique weak solution of the PDE (1.8).

The definition of our dynamics implies that for all $\varphi \in \mathcal{C}^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$, we have

$$\begin{aligned} \langle m^N(t), \varphi \rangle_N &= \langle m^N(0), \varphi \rangle_N + \frac{1}{2} \int_0^t \langle m^N(s), (2N)^2 \Delta \varphi \rangle_N ds \\ &\quad + \mathcal{O}(N^{1-\alpha}) + M_t^N(\varphi) , \end{aligned} \quad (4.12)$$

where $M^N(\varphi)$ is a martingale with bracket

$$\langle M^N(\varphi) \rangle_t = \int_0^t \frac{4}{2N} \langle \varphi^2, p_N \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) > 0\}} + (1 - p_N) \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) < 0\}} \rangle_N ds$$

$$\leq \frac{4t\|\varphi\|_\infty^2}{2N}.$$

The jumps of $M^N(\varphi)$ are almost surely bounded by a term of order $1/2N$, uniformly over all $N \geq 1$. Then, by (6.3) we get

$$\mathbb{E}^N \left[\sup_{t \leq T} |M_t^N(\varphi)|^2 \right]^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{2N}} + \frac{1}{2N},$$

uniformly over all $N \geq 1$, so that $M^N(\varphi)$ vanishes in probability as $N \rightarrow \infty$. Then classical arguments ensure along a converging subsequence of m^N , we can pass to the limit on (4.12) and get (4.11), thus concluding the proof. \square

Proof of Theorem 1.5, $\alpha = 1$. In that case, we characterise the limit via the Hopf-Cole transform $\xi(t, x) = \exp(-2\sigma m(t, x) + 2\sigma^2 t)$ that maps, formally, the PDE (1.9) into

$$\begin{cases} \partial_t \xi = \frac{1}{2} \partial_x^2 \xi, & x \in [0, 1], \quad t > 0, \\ \xi(t, 0) = \xi(t, 1) = e^{2\sigma^2 t}, & \xi(0, \cdot) = e^{-2\sigma m(0, \cdot)}. \end{cases} \quad (4.13)$$

This equation admits a unique weak solution in the space of continuous space-time functions, and it is well-known that the unique weak solution of (1.9) coincides with the latter solution upon reverse Hopf-Cole transform.

A famous result due to Gärtner [Gär88] shows that a similar transform, performed at the level of the exclusion process, linearises the drift of the stochastic differential equations solved by our discrete process. Namely, if one sets

$$\gamma_N = \frac{2\sigma}{2N}, \quad c_N = \frac{(2N)^2}{e^{\gamma_N} + e^{-\gamma_N}}, \quad \lambda_N = c_N(e^{\gamma_N} - 2 + e^{-\gamma_N}),$$

and

$$\xi^N(t, x) := e^{-\gamma_N S(t(2N)^2, 2Nx) + \lambda_N t}, \quad x \in [0, 1], \quad t \geq 0,$$

then, using the abusive notation $\xi^N(t, k)$ for $\xi^N(t, x)$ when $x = k/2N$, we have

$$\begin{cases} d\xi^N(t, k) = c_N \Delta \xi^N(t, k) dt + d\tilde{M}^N(t, k), \\ \xi^N(t, 0) = \xi^N(t, 1) = e^{\lambda_N t}, \end{cases}$$

where Δ is the discrete Laplacian and $\tilde{M}^N(t, k)$ is a martingale with quadratic variation given by

$$\begin{aligned} \langle \tilde{M}^N(\cdot, k) \rangle_t &= (2N)^2 \int_0^t \xi^N(s, k)^2 \left((e^{-2\gamma_N} - 1)^2 \mathbf{1}_{\{\Delta S(s(2N)^2, k) > 0\}} p_N \right. \\ &\quad \left. + (e^{2\gamma_N} - 1)^2 \mathbf{1}_{\{\Delta S(s(2N)^2, k) < 0\}} (1 - p_N) \right) ds. \end{aligned} \quad (4.14)$$

The tightness of m^N easily implies the tightness of ξ^N . It only remains to identify the limit. To that end, we observe that for all $\varphi \in C^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$, we have

$$\begin{aligned} \langle \xi^N(t), \varphi \rangle_N &= \langle \xi^N(0), \varphi \rangle_N + R_t^N(\varphi) \\ &\quad + c_N \int_0^t \left(\langle \xi^N(s), \Delta \varphi \rangle_N + \frac{1}{2N} e^{\lambda_N s} \left(\varphi\left(\frac{1}{2N}\right) + \varphi\left(\frac{2N-1}{2N}\right) \right) \right) ds, \end{aligned}$$

where

$$R_t^N(\varphi) = \int_0^t \frac{1}{2N} \sum_{k=1}^{2N-1} \varphi\left(\frac{k}{2N}\right) d\tilde{M}^N(s, k) .$$

It is elementary to check that there exists $C > 0$ such that $|\xi^N(t, k)| \leq C$ for all t in a compact set of \mathbf{R}_+ , all $k \in \{1, \dots, 2N-1\}$ and all $N \geq 1$. Consequently there exists $C' > 0$ such that $\langle \tilde{M}^N(\cdot, k) \rangle_t \leq C't$ uniformly over the same set of parameters. Moreover, the jumps of this martingale are uniformly bounded by some constant on the same set of parameters. Then, a simple calculation based on (6.3) shows that for any given $t \geq 0$, the moments of $R_t^N(\varphi)$ vanish as $N \rightarrow \infty$. Hence, any limit ξ of a converging subsequence of ξ^N satisfies

$$\langle \xi(t), \varphi \rangle = \langle \xi(0), \varphi \rangle + \frac{1}{2} \int_0^t \left(\langle \xi(s), \varphi'' \rangle + e^{2\sigma^2 s} (\varphi'(0) - \varphi'(1)) \right) ds ,$$

for all φ as above, and therefore coincides with the unique weak solution of (4.13), thus concluding the proof. \square

4.3 Proof of Theorem 1.6

Regarding the process v^N , the arguments of the tightness are slightly more involved. However, the strategy of proof is the same as in the previous subsection. First of all, we introduce a time-interpolation \bar{v}^N of v^N and the same arguments as before yield an analogous result as Lemma 4.5. The stochastic differential equations solved by the discrete process v^N are given by

$$\begin{aligned} dv^N(t, \ell) &= \frac{(2N)^2}{2} \Delta v^N(t, \ell) dt \\ &\quad + (2p_N - 1)(2N)^\alpha \mathbf{1}_{\{\Delta S(t(2N)^2, \ell) \neq 0\}} dt + dM^N(t, \ell) , \end{aligned}$$

where M^N is a martingale with bracket given by

$$d\langle M^N(\cdot, \ell) \rangle_t = \frac{4(2N)^2}{(2N)^{4-2\alpha}} \left(p_N \mathbf{1}_{\{\Delta S(t(2N)^2, \ell) > 0\}} + (1 - p_N) \mathbf{1}_{\{\Delta S(t(2N)^2, \ell) < 0\}} \right) dt .$$

If we let $p_t^N(k, \ell)$ be the fundamental solution of the discrete heat equation, see (6.4) with $c_N = (2N)^2/2$, then it is simple to check that we have

$$\begin{aligned} v^N(t, \ell) &= \sum_k p_t^N(k, \ell) v^N(0, k) + N_t^t(\ell) \\ &\quad + (2p_N - 1)(2N)^\alpha \int_0^t \sum_k p_{t-s}^N(k, \ell) \mathbf{1}_{\{\Delta S(s(2N)^2, k) \neq 0\}} ds , \end{aligned} \tag{4.15}$$

where $N_s^t(\ell)$ is the martingale defined by

$$N_s^t(\ell) := \int_0^s \sum_k p_{t-s}^N(k, \ell) dM^N(s, k) , \quad s \in [0, t] .$$

Recall that $\delta_i \in (0, 1)$ is the Hölder regularity of the initial condition.

Lemma 4.8 *For all $\delta \in (0, \frac{\delta_i}{2} \wedge \frac{1}{4})$, all $T > 0$ and all $p \geq 1$, we have*

$$\mathbb{E}^N \left[|v^N(t', x) - v^N(t, x)|^p \right]^{\frac{1}{p}} \lesssim |t' - t|^\delta + \frac{1}{(2N)^{2-\alpha}},$$

uniformly over all $t', t \in [0, T]$, all $x \in [0, 1]$ and all $N \geq 1$.

The proof of this result can be obtained from that of Lemma 4.7 up to some simple modifications. On the other hand, the space regularity of our discrete process needs to be controlled and this is the content of the following lemma.

Lemma 4.9 *For all $\delta \in (0, \delta_i \wedge \frac{1}{2})$, all $T > 0$ and all $p \geq 1$, we have*

$$\mathbb{E}^N \left[|v^N(t, x) - v^N(t, y)|^p \right]^{\frac{1}{p}} \lesssim |x - y|^\delta,$$

uniformly over all $t \in [0, T]$, all $x, y \in [0, 1]$ and all $N \geq 1$.

Proof. It suffices to establish the bound for x, y of the form $\ell/2N, \ell'/2N$ since the remaining cases follow by interpolation. Given the expression (4.15), the increment $v^N(t, \ell) - v^N(t, \ell')$ can be written as the sum of three terms: the contribution of the initial condition, of the asymmetry and of the martingale terms. We bound separately the p -th moments of these three terms.

Similarly as in the proof of Lemma 4.7, we use the fact that $v^N(0, \cdot)$ is δ -Hölder uniformly in $N \geq 1$ to deduce that

$$\left| \sum_{k=1}^{2N-1} (p_t^N(k, \ell) - p_t^N(k, \ell')) v^N(0, k) \right| \lesssim \left(\frac{|\ell - \ell'|}{2N} \right)^\delta,$$

uniformly over all $N \geq 1$, all ℓ, ℓ' and all $t \geq 0$, as required.

We turn to the contribution of the asymmetry. Using classical estimates on the heat kernel, recalled in Appendix 6.2 we have almost surely

$$\begin{aligned} & \left| \int_0^t \sum_k (p_{t-s}^N(k, \ell) - p_{t-s}^N(k, \ell')) \mathbf{1}_{\{\Delta S(t(2N)^2, k) \neq 0\}} ds \right| \\ & \leq \int_0^t \sum_k |p_{t-s}^N(k, \ell) - p_{t-s}^N(k, \ell')| ds \lesssim \left(\frac{|\ell - \ell'|}{2N} \right)^\delta, \end{aligned}$$

uniformly over all $N \geq 1$, all $t \in [0, T]$ and all $\ell, \ell' \in \{1, \dots, 2N-1\}$, thus yielding the desired bound.

Finally, we treat the martingale term. We aim at applying (6.3) to the martingale $s \mapsto N_s^t(\ell) - N_s^t(\ell')$ at time t . First, observe that the absolute value of the jumps of this martingale are bounded by $2/(2N)^{2-\alpha}$. Second, we have the following almost sure bound

$$\begin{aligned} \langle N^t(\ell) - N^t(\ell') \rangle_t & \leq \frac{4}{(2N)^{2-2\alpha}} \int_0^t \sum_{k=1}^{2N-1} |p_{t-s}^N(k, \ell) - p_{t-s}^N(k, \ell')|^2 ds \\ & \lesssim \frac{1}{(2N)^{2-2\alpha}} \int_0^t \frac{1}{2N\sqrt{t-s}} \left(\frac{|\ell - \ell'|}{2N} \right)^{2\delta} ds \end{aligned}$$

$$\lesssim \frac{1}{(2N)^{3-2\alpha}} \left(\frac{|\ell - \ell'|}{2N} \right)^{2\delta},$$

uniformly over all $N \geq 1$, all $\ell, \ell' \in \{1, \dots, 2N - 1\}$ and all $t \in [0, T]$. Applying (6.3), we get a bound of order

$$\frac{1}{(2N)^{\frac{3-2\alpha}{2}}} \left(\frac{|\ell - \ell'|}{2N} \right)^{\delta} + \frac{1}{(2N)^{2-\alpha}},$$

for the p -th moment of $N_t^t(\ell) - N_t^t(\ell')$. As soon as $\ell \neq \ell'$, we find $(2N)^{-(2-\alpha)} \lesssim (|\ell - \ell'|/(2N))^{\delta}$ thus concluding the proof. \square

From there, the arguments presented for m^N apply verbatim to v^N and show tightness of the sequence. Let v be the limit of a converging subsequence. To conclude the proof, we only need to show that for any $\varphi \in \mathcal{C}^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$, we have

$$\langle v(t), \varphi \rangle = \langle v(0), \varphi \rangle + \frac{1}{2} \int_0^t \langle v(s), \varphi'' \rangle ds + \sigma t \langle 1, \varphi \rangle. \quad (4.16)$$

This suffices to identify the unique weak solution of the PDE (1.13).

The definition of our dynamics implies that for all $\varphi \in \mathcal{C}^2([0, 1])$ such that $\varphi(0) = \varphi(1) = 0$, we have

$$\begin{aligned} \langle v^N(t), \varphi \rangle_N &= \langle v^N(0), \varphi \rangle_N + \frac{1}{2} \int_0^t \langle v^N(s), (2N)^2 \Delta \varphi \rangle_N ds \\ &\quad + (2p_N - 1)(2N)^{\alpha} \int_0^t \left\langle \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) \neq 0\}}, \varphi \right\rangle_N ds + M_t^N(\varphi), \end{aligned} \quad (4.17)$$

where $M^N(\varphi)$ is a martingale with bracket

$$\begin{aligned} \langle M^N(\varphi) \rangle_t &= \int_0^t \frac{4}{(2N)^{3-2\alpha}} \langle \varphi^2, p_N \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) > 0\}} + (1 - p_N) \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) < 0\}} \rangle_N ds \\ &\leq \frac{4t \|\varphi\|_{\infty}^2}{(2N)^{3-2\alpha}}. \end{aligned}$$

The jumps of $M^N(\varphi)$ are almost surely bounded by a term of order $1/(2N)^{3-2\alpha}$, uniformly over all $N \geq 1$. Then, by (6.3) we get

$$\mathbb{E}^N \left[\sup_{t \leq T} |M_t^N(\varphi)|^2 \right]^{\frac{1}{2}} \lesssim \frac{1}{(2N)^{\frac{3}{2}-\alpha}} + \frac{1}{(2N)^{3-2\alpha}},$$

uniformly over all $N \geq 1$, so that $M^N(\varphi)$ vanishes in probability as $N \rightarrow \infty$.

One would like to pass to the limit on (4.17) along a converging subsequence: by classical arguments, we get all the terms in (4.16) except the last term on the right which needs some additional care. More precisely, we will show that the term in (4.12) that contains an indicator, converges in probability to the last term on the right of (4.16). To that end, we apply Theorem 4.2 with $\ell = 2$ and $\Phi(\eta) = \eta(1)(1 - \eta(2)) + (1 - \eta(1))\eta(2)$. Recall the one-to-one correspondence between particle configurations $\eta \in \{0, 1\}^{2N}$ and discrete paths $S(n), n \in [0, 2N]$:

given this correspondence, $\Phi(\eta)$ is the indicator of the event $\{\Delta S(1) \neq 0\}$, and $\tilde{\Phi}(a) = 2a(1 - a)$. Then, we observe that

$$\left\langle \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) \neq 0\}}, \varphi \right\rangle_N = \frac{1}{2N} \sum_{k=1}^{2N} \varphi\left(\frac{k}{2N}\right) M_{T_{\epsilon N}(k)} \Phi(\eta_s) + \mathcal{O}(\epsilon),$$

where $\mathcal{O}(\epsilon)$ vanishes as $\epsilon \downarrow 0$. This being given, we deduce that we can replace

$$\int_0^t \left\langle \mathbf{1}_{\{\Delta S(s(2N)^2, \cdot) \neq 0\}}, \varphi \right\rangle_N ds$$

by

$$\int_0^t \frac{1}{2N} \sum_{k=1}^{2N} \tilde{\Phi}\left(M_{T_{\epsilon N}(k)} \eta_s\right) \varphi\left(\frac{k}{2N}\right) ds, \quad (4.18)$$

up to an error of order

$$t\epsilon + \frac{1}{2N} \int_0^t \sum_{k=1}^{2N} V_{\epsilon N}(\tau_k \eta_s) ds.$$

By Theorem 4.2, this error term goes to 0 in probability as $N \rightarrow \infty$ and $\epsilon \downarrow 0$. Using the tightness of the sequence v^N , we deduce that (4.18) converges to $t\langle 1, \varphi \rangle / 2$ in probability as $N \rightarrow \infty$ and $\epsilon \downarrow 0$. Since $(2p_N - 1)(2N)^\alpha$ goes to 2σ as $N \rightarrow \infty$, we deduce that the third term on the right hand side of (4.17) converges in probability to $\sigma t \langle 1, \varphi \rangle$ as $N \rightarrow \infty$, thus concluding the proof.

4.4 Hydrodynamic limit: the hyperbolic case*

Weak solutions of (1.11) are not unique, therefore we need a criterion to pick the physical solution and this is provided by the so-called entropy inequalities.

Definition 4.10 Let $\eta_0 \in L^\infty(0, 1)$. We say that $\eta \in L^\infty((0, \infty) \times (0, 1))$ is an entropy solution of (1.11) if:

- For all $c \in [0, 1]$ and all $\varphi \in \mathcal{C}_c^\infty((0, \infty) \times (0, 1), \mathbf{R}_+)$, we have

$$\begin{aligned} \int_0^\infty \int_0^1 & \left(|\eta(t, x) - c| \partial_t \varphi(t, x) - 2 \operatorname{sgn}(\eta(t, x) - c) \right. \\ & \left. \times ((\eta(t, x)(1 - \eta(t, x)) - c(1 - c)) \partial_x \varphi(t, x)) \right) dx dt \geq 0, \end{aligned}$$

- We have $\operatorname{esslim}_{t \downarrow 0} \int_0^1 |\eta(t, x) - \eta_0(x)| dx = 0$,
- We have $\eta(t, x)(1 - \eta(t, x)) = 0$ for almost all $t > 0$ and all $x \in \{0, 1\}$.

Let us mention that the first condition is sufficient to ensure that η has a trace at the boundaries so that the third condition is meaningful. Bürger, Frid and Karlsen [BFK07] show existence and uniqueness of entropy solutions with zero-flux boundary condition.

*This section is taken from [Lab16]

Let us now introduce the inviscid Burgers equation with some appropriate Dirichlet boundary conditions:

$$\begin{cases} \partial_t \eta = 2\partial_x(\eta(1-\eta)) , \\ \eta(t, 0) = 1 , \quad \eta(t, 1) = 0 , \\ \eta(0, \cdot) = \eta_0(\cdot) . \end{cases} \quad (4.19)$$

The precise definition of the entropy solution of (4.19) is the same as Definition 4.10 except for the third condition which must be replaced by the so-called BLN conditions

$$\begin{aligned} \operatorname{sgn}(\eta(t, 0) - 1)(\eta(t, 0)(1 - \eta(t, 0)) - c(1 - c)) &\geq 0 , \quad \forall c \in [\eta(t, 0), 1] \\ \operatorname{sgn}(\eta(t, 1) - 0)(\eta(t, 1)(1 - \eta(t, 1)) - c(1 - c)) &\leq 0 , \quad \forall c \in [0, \eta(t, 1)] , \end{aligned} \quad (4.20)$$

for almost all $t > 0$. Here again, there is existence and uniqueness of entropy solutions of (4.19).

Proposition 4.11 *The entropy solutions of (1.11) and (4.19) coincide.*

Proof. Both solutions exist and are unique. Let us show that the solution of (4.19) satisfies the conditions of Definition 4.10: actually, the two first conditions are automatically satisfied, so we focus on the third one. The BLN conditions above immediately imply that $\eta(t, 0)$ and $\eta(t, 1)$ are necessarily in $\{0, 1\}$ for almost all $t > 0$, so that the third condition is satisfied. \square

As a consequence, we can choose the formulation (4.19) in the proof of our convergence result. We will rely on some properties of the solution that we now recall: we refer to Vovelle [Vov02] for the first part of the statement, and to Bürger, Frid and Karlsen [BFK07] for the second part.

Proposition 4.12 *Let $\eta_0 \in L^\infty(0, 1)$. A function $\eta \in L^\infty((0, \infty) \times (0, 1))$ is the entropy solution of (4.19) if and only if for all $c \in [0, 1]$ and all $\varphi \in \mathcal{C}_c^\infty([0, \infty) \times [0, 1], \mathbf{R}_+)$ we have*

$$\begin{aligned} &\int_0^\infty \int_0^1 ((\eta(t, x) - c)^\pm \partial_t \varphi(t, x) + h^\pm(\eta(t, x), c) \partial_x \varphi(t, x)) dx dt \\ &+ \int_0^1 (\eta_0(x) - c)^\pm \varphi(0, x) dx + 2 \int_0^\infty \left((1 - c)^\pm \varphi(t, 0) + (0 - c)^\pm \varphi(t, 1) \right) dt \geq 0 , \end{aligned} \quad (4.21)$$

where $(x)^\pm$ denotes the positive/negative part of $x \in \mathbf{R}$, $\operatorname{sgn}^\pm(x) = \pm \mathbf{1}_{(0, \infty)}(\pm x)$ and $h^\pm(\eta, c) := -2 \operatorname{sgn}^\pm(\eta - c)(\eta(1 - \eta) - c(1 - c))$.

Furthermore for any $t > 0$, the map $\eta_0 \mapsto \eta(t)$ is 1-Lipschitz in $L^1(0, 1)$.

Now that we have presented the precise meaning given to the asserted limit, we proceed to the proof of the convergence. We let $\mathcal{M}([0, 1])$ be the space of finite measures on $[0, 1]$ endowed with the topology of weak convergence. Recall the process ϱ^N defined in (1.12).

Proposition 4.13 *Let ν_N be any measure on $\{0, 1\}^{2N}$. The sequence of processes $(\varrho_t^N, t \geq 0)$, starting from ν_N , is tight in the space $\mathbb{D}([0, \infty), \mathcal{M}([0, 1]))$. Furthermore, the associated sequence of processes $(m^N(t, x), t \geq 0, x \in [0, 1])$ is tight in $\mathbb{D}([0, \infty), \mathcal{C}([0, 1]))$.*

Note that for a generic measure ι_N on $\{0, 1\}^{2N}$, $m^N(t, 1)$ is not necessarily equal to 0.

Proof. Let $\varphi \in \mathcal{C}^2([0, 1])$. It suffices to show that $\langle \varrho_0^N, \varphi \rangle$ is tight in \mathbf{R} , and that for all $T > 0$

$$\lim_{h \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\iota_N}^N \left[\sup_{s, t \leq T, |t-s| \leq h} |\langle \varrho_t^N - \varrho_s^N, \varphi \rangle| \right] = 0. \quad (4.22)$$

The former is immediate since $|\langle \varrho_0^N, \varphi \rangle| \leq \|\varphi\|_\infty$. Regarding the latter, we let L^N be the generator of our sped-up process and we write

$$\langle \varrho_t^N - \varrho_s^N, \varphi \rangle = \frac{1}{2N} \int_s^t \sum_{k=1}^{2N} \varphi(k) L^N \eta_r(k) dr + M_{s,t}^N(\varphi),$$

where $M_{s,t}^N(\varphi)$ is a martingale. Its bracket can be bounded almost surely as follows

$$\langle M_{s,t}^N(\varphi) \rangle_t \leq \int_s^t \frac{1}{(2N)^2} \sum_{k=1}^{2N-1} (\nabla \varphi(k))^2 (2N)^{1+\alpha} dr \lesssim \frac{t-s}{(2N)^{2-\alpha}}.$$

Since the jumps of this martingale are bounded by a term of order $\|\varphi'\|_\infty / (2N)^2$, the BDG inequality (6.3) ensures that the expectation of $\sup_{t \in [s, s+h]} |M_{s,t}^N(\varphi)|$ vanishes as $N \rightarrow \infty$. Consequently,

$$\lim_{h \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{h} \sup_{s \in [0, T]} \mathbb{E}_{\iota_N}^N \left[\sup_{t \leq T, |t-s| \leq h} |M_{s,t}^N(\varphi)| \right] = 0. \quad (4.23)$$

Let us bound the term involving the generator. Decomposing the jump rates into the symmetric part (of intensity $1 - p_N$) and the totally asymmetric part (of intensity $2p_N - 1$), we find

$$\begin{aligned} \frac{1}{2N} \sum_{k=1}^{2N} \varphi(k) \mathcal{L}^N \eta(k) &= -(2N)^\alpha (1 - p_N) \sum_{k=1}^{2N-1} \nabla \eta(k) \nabla \varphi(k) \\ &\quad - (2N)^\alpha (2p_N - 1) \sum_{k=1}^{2N-1} \eta(k+1) (1 - \eta(k)) \nabla \varphi(k). \end{aligned}$$

A simple integration by parts shows that the first term on the right is bounded by a term of order $N^{\alpha-1}$ while the second term is of order 1. Consequently

$$\mathbb{E}_{\iota_N}^N \left[\sup_{s, t \leq T, |t-s| \leq h} \left| \frac{1}{2N} \int_s^t \sum_{k=1}^{2N} \varphi(k) L^N \eta_r(k) dr \right| \right] \lesssim h, \quad (4.24)$$

uniformly over all $N \geq 1$ and all $h > 0$. The l.h.s. vanishes as $N \rightarrow \infty$ and $h \downarrow 0$. Combining (4.23) and (4.24), (4.22) follows.

We turn to the tightness of the interface m^N . First, the profile $m^N(t, \cdot)$ is 1-Lipschitz for all $t \geq 0$ and all $N \geq 1$. Second, we claim that for some $\beta \in (\alpha, 1)$

$$\mathbb{E}_{\iota_N}^N \left[|m^N(t, k) - m^N(s, k)|^p \right]^{\frac{1}{p}} \lesssim |t-s| + \frac{1}{N^{1-\beta}}, \quad (4.25)$$

uniformly over all $0 \leq s \leq t \leq T$, all $k \in \{1, \dots, 2N\}$ and all $N \geq 1$. This being given, the arguments for proving tightness are classical: one introduces a piecewise linear time-interpolation \bar{m}^N of m^N and shows tightness for this process, and then one shows that the difference between \bar{m}^N and m^N is uniformly small. We are left with the proof of (4.25). Let $\psi : \mathbf{R} \rightarrow \mathbf{R}_+$ be a non-increasing, smooth function such that $\psi(x) = 1$ for all $x \leq 0$ and $\psi(x) = 0$ for all $x \geq 1$. Fix $\beta \in (\alpha, 1)$. For any given $k \in \{1, \dots, 2N\}$, we define $\varphi_k^N : \{0, \dots, 2N\} \rightarrow \mathbf{R}$ by setting $\varphi_k^N(\ell) = \psi((\ell - k)/(2N)^\beta)$. Then, we observe that

$$\frac{1}{2N} \sum_{\ell=1}^{2N} (2\eta_t(\ell) - 1) \varphi_k^N(\ell) = m^N(t, k) + \mathcal{O}(N^{\beta-1}),$$

uniformly over all $k \in \{1, \dots, 2N\}$ and all $t \geq 0$. Then, similar computations to those made in the first part of the proof show that

$$\mathbb{E}_{\iota_N}^N \left[\left| \frac{1}{2N} \sum_{\ell=1}^{2N} (\eta_t(\ell) - \eta_s(\ell)) \varphi_k^N(\ell) \right|^p \right]^{\frac{1}{p}} \lesssim (t - s) + \sqrt{\frac{t - s}{N^{1+\beta-\alpha}}} + \frac{1}{N^{1+\beta}},$$

uniformly over all k , all $0 \leq s \leq t \leq T$ and all $N \geq 1$. This yields (4.25). \square

The main step in the proof of Theorem 1.5 is to prove the convergence of the density of particles, starting from a product measure satisfying Assumption 4.1. Under this assumption and if the process starts from ι_N , then ϱ_0^N converges to the deterministic limit $\varrho_0(dx) = f(x)dx$.

Theorem 4.14 *Under Assumption 4.1, the process ϱ^N converges in distribution in the Skorohod space $\mathbb{D}([0, \infty), \mathcal{M}([0, 1]))$ to the deterministic process $(\eta(t, x)dx, t \geq 0)$, where η is the entropy solution of Proposition 4.12 starting from f .*

Given this result, the proof of the hydrodynamic limit is simple.

Proof of Theorem 1.5. Let ι_N be as in Theorem 4.14. We know that ϱ^N converges to some limit ϱ and that m^N is tight. Let m be some limit point and let m^{N_i} be an associated converging subsequence. By Skorohod's representation theorem, we can assume that (ϱ^{N_i}, m^{N_i}) converges almost surely to (ϱ, m) . Recall that ϱ is of the form $\varrho(t, x) = \eta(t, x)dx$. Our first goal is to show that $m(t, x) = \int_0^x (2\eta(t, y) - 1)dy$ for all t, x .

Fix $x_0 \in (0, 1)$. We introduce an approximation of the indicator of $(-\infty, x_0]$ by setting $\varphi_p(\cdot) = 1 - \int_{-\infty}^\cdot P_{1/p}(y - x_0)dy$, $p \geq 1$ where P_t is the heat kernel on \mathbf{R} at time t . For each $p \geq 1$, φ_p is smooth and for any $\delta > 0$ we have

$$\|\varphi_p - \mathbf{1}_{[0, x_0]}\|_{L^1(0, 1)} \rightarrow 0, \quad \sup_{f \in \mathcal{C}^\delta([0, 1])} \frac{|\langle f, \delta_{x_0} + \partial_x \varphi_p \rangle|}{\|f\|_{\mathcal{C}^\delta}} \rightarrow 0, \quad (4.26)$$

as $p \rightarrow \infty$. If we set $I(t, x_0) = m(t, x_0) - \int_0^{x_0} (2\eta(t, y) - 1)dy$ for some $x_0 \in (0, 1)$ and some $t > 0$, then $|I(t, x_0)|$ is bounded by

$$\begin{aligned} & \|m(t) - m^N(t)\|_\infty + |\langle m^N(t), \delta_{x_0} + \partial_x \varphi_p \rangle| + |\langle m^N(t), \partial_x \varphi_p \rangle + \langle 2\varrho_t^N - 1, \varphi_p \rangle| \\ & + 2|\langle \varrho_t^N - \varrho_t, \varphi_p \rangle| + |\langle 2\varrho(t) - 1, \varphi_p - \mathbf{1}_{[0, x_0]} \rangle|. \end{aligned}$$

Recall that m^N is 1-Lipschitz in space, so that the second term vanishes as $p \rightarrow \infty$ by (4.26). A discrete integration by parts shows that the third term vanishes as N goes to ∞ . The first and fourth terms vanish as $N \rightarrow \infty$ by the convergence of m^N and ϱ^N , and the last term is dealt with using (4.26). Choosing p and then N large enough, we deduce that $|I(t, x_0)|$ is almost surely as small as desired. This identifies completely the limit m of any converging subsequence, under $\mathbb{P}_{t_N}^N$.

We are left with the extension of this convergence result to an arbitrary initial condition. Assume that $m^N(0, \cdot)$ converges to some profile $m_0(\cdot)$ for the supremum norm. Since each $m^N(0, \cdot)$ is 1-Lipschitz, so is m_0 . Let $n \geq 1$ and $\epsilon > 0$. One can find two profiles $m_0^{n,\epsilon,+}$ and $m_0^{n,\epsilon,-}$ which are 1-Lipschitz, affine on each interval $[k/n, (k+1)/n)$ with $k \in \{0, \dots, n-1\}$, start from 0 and are such that $m_0^{n,\epsilon,-} \leq (m_0 - \epsilon) \vee (-x) \vee (x-1)$ and $(m_0 + \epsilon) \wedge x \wedge (1-x) \leq m_0^{n,\epsilon,+}$. We define a coupling of three instances of our height process $(m^{N,n,\epsilon,-}, m^N, m^{N,n,\epsilon,+})$ which preserves the order of the interfaces and is such that $m^{N,n,\epsilon,\pm}(0, \cdot)$ is the height function associated with the particle density distributed as $\otimes_{k=1}^{2N} \text{Be}(\varrho^{n,\epsilon,\pm}(k/2N))$ where $\varrho^{n,\epsilon,\pm}(\cdot) = (\partial_x m_0^{n,\epsilon,\pm}(\cdot) + 1)/2$. It is simple to check that the probability of the event $m^{N,n,\epsilon,-}(0, \cdot) \leq m^N(0, \cdot) \leq m^{N,n,\epsilon,+}(0, \cdot)$ goes to 1 as $N \rightarrow \infty$. Our convergence result applies to $m^{N,n,\epsilon,\pm}$ and, consequently, any limit point of the tight sequence m^N is squeezed in $[m^{n,\epsilon,-}, m^{n,\epsilon,+}]$ where $m^{n,\epsilon,\pm}$ is the entropy solution of (1.11) starting from $m_0^{n,\epsilon,\pm}$. By the L^1 -contractivity of the solution map associated with (1.11), see [MNRR96, Thm.7.28], we deduce that $m^{n,\epsilon,\pm}$ converge, as $n \rightarrow \infty$ and $\epsilon \downarrow 0$, to the integrated entropy solution of (1.11) starting from m_0 , thus concluding the proof. \square

To prove Theorem 4.14, we need to show that the limit of any converging subsequence of ϱ^N is of the form $\varrho(t, dx) = \eta(t, x)dx$ and that η satisfies the entropy inequalities of Proposition 4.12. To make appear the constant c in these inequalities, the usual trick is to define a coupling of the particle system η^N with another particle system ζ^N which is stationary with density c so that, at large scales, one can replace the averages of ζ^N by c . Such a coupling has been defined by Rezakhanlou [Rez91] in the case of the infinite lattice \mathbb{Z} . The specificity of the present setting comes from the boundary conditions of our system: one needs to choose carefully the flux of particles at 1 and $2N$ for ζ^N .

The precise definition of our coupling goes as follows. We set

$$p(1) = 1 - p_N, \quad p(-1) = p_N, \quad \text{and} \quad p(k) = 0 \quad \forall k \neq \{-1, 1\},$$

as well as $b(a, a') = a(1 - a')$. Then, we define

$$\begin{aligned} \tilde{\mathcal{L}}^{\text{bulk}} f(\eta, \zeta) = (2N)^{1+\alpha} \sum_{k,\ell=1}^{2N} p(\ell - k) \times \\ \left[(b(\eta(k), \eta(\ell)) \wedge b(\zeta(k), \zeta(\ell))) (f(\eta^{k,\ell}, \zeta^{k,\ell}) - f(\eta, \zeta)) \right. \\ + (b(\eta(k), \eta(\ell)) - b(\eta(k), \eta(\ell)) \wedge b(\zeta(k), \zeta(\ell))) (f(\eta^{k,\ell}, \zeta) - f(\eta, \zeta)) \\ \left. + (b(\zeta(k), \zeta(\ell)) - b(\eta(k), \eta(\ell)) \wedge b(\zeta(k), \zeta(\ell))) (f(\eta, \zeta^{k,\ell}) - f(\eta, \zeta)) \right], \end{aligned}$$

and, using the notation $\zeta \pm \delta_k$ to denote the particle configuration which coincides with ζ everywhere except at site k where the occupation is $\zeta(k) \pm 1$,

$$\tilde{\mathcal{L}}^{\text{bdry}} f(\eta, \zeta) = (2N)^{1+\alpha} (2p_N - 1)(1 - c)\zeta(1)(f(\eta, \zeta - \delta_1) - f(\eta, \zeta))$$

$$+ (2N)^{1+\alpha} (2p_N - 1) c (1 - \zeta(2N)) (f(\eta, \zeta + \delta_{2N}) - f(\eta, \zeta)) .$$

We consider the stochastic process $(\eta_t^N, \zeta_t^N), t \geq 0$ associated to the generator $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^{\text{bulk}} + \tilde{\mathcal{L}}^{\text{bdry}}$. From now on, we will always assume that η_0^N has law ι_N , where ι_N satisfies Assumption 4.1, and that ζ_0^N is distributed as a product of Bernoulli measures with parameter c . Furthermore, we will always assume that the coupling at time 0 is such that

$$\text{sgn}(\eta_0^N(k) - \zeta_0^N(k)) = \text{sgn}(f(k/2N) - c), \quad \forall k \in \{1, \dots, 2N\},$$

where f is the macroscopic density profile of Assumption 4.1. Notice that this is always possible to construct such a coupling. We let $\tilde{\mathbb{P}}_{\iota_N, c}^N$ be the law of the process (η^N, ζ^N) .

Remark 4.15 The bulk part of the generator prescribes the dynamics of the simple exclusion process for both particle systems. The coupling is such that the order of ζ^N and η^N is preserved: this is the reason why the expression of the generator is non-trivial. The boundary part of the generator prescribes the minimal jump rates for ζ^N to be stationary with density c : there is neither entering flux at 1 nor exiting flux at $2N$. This choice is convenient for establishing the entropy inequalities.

It will actually be important to track the sign changes in the pair (η^N, ζ^N) . To that end, we let $F_{k,\ell}(\eta, \zeta) = 1$ if $\eta(k) \geq \zeta(k)$ and $\eta(\ell) \geq \zeta(\ell)$; and $F_{k,\ell}(\eta, \zeta) = 0$ otherwise. We say that a subset C of consecutive integers in $\{1, \dots, 2N\}$ is a cluster with constant sign if for all $k, \ell \in C$ we have $F_{k,\ell}(\eta, \zeta) = 1$, or for all $k, \ell \in C$ we have $F_{k,\ell}(\zeta, \eta) = 1$. For a given configuration (η, ζ) , we let n be the minimal number of clusters needed to cover $\{1, \dots, 2N\}$: we will call n the number of sign changes. There is not necessarily a unique choice of covering into n clusters. Let $C(i), i \leq n$ be any such covering and let $1 = k_1 < k_2 < \dots < k_n < k_{n+1} = 2N + 1$ be the integers such that $C(i) = \{k_i, k_{i+1} - 1\}$.

Lemma 4.16 *Under $\tilde{\mathbb{P}}_{\iota_N, c}^N$, the process η^N has law $\mathbb{P}_{\iota_N}^N$ while the process ζ^N is stationary with law $\otimes_{k=1}^{2N} \text{Be}(c)$. Furthermore, the number of sign changes $n(t)$ is smaller than $n(0) + 3$ at all time $t \geq 0$.*

Proof. It is simple to check the assertion on the laws of the marginals η^N and ζ^N . Regarding the number of sign changes, the key observation is the following. In the bulk $\{2, \dots, 2N - 1\}$, to create a new sign change we need to have two consecutive sites k, ℓ such that $\eta^N(k) = \zeta^N(k) = 1, \eta^N(\ell) = \zeta^N(\ell) = 0$ and we need to let one particle jump from k to ℓ , but not both. However, our coupling does never allow such a jump. Therefore, the number of sign changes can only increase at the boundaries due to the interaction of ζ^N with the reservoirs: this can create at most 2 new sign changes, thus concluding the proof. \square

Assumption 4.1 ensures the existence of a constant $C > 0$ such that $n(0) < C$ almost surely for all $N \geq 1$. We now derive the entropy inequalities at the microscopic level. Recall that τ_k stands for the shift operator with periodic boundary conditions, and let $\langle u, v \rangle_N = (2N)^{-1} \sum_{k=1}^{2N} u(k/2N) v(k/2N)$ denote the discrete L^2 product.

Lemma 4.17 (Microscopic inequalities) *Let ι_N be a measure on $\{0, 1\}^{2N}$ satisfying Assumption 4.1. For all $\varphi \in \mathcal{C}_c^\infty([0, \infty) \times [0, 1], \mathbf{R}_+)$, all $\delta > 0$ and all $c \in [0, 1]$, we have $\lim_{N \rightarrow \infty} \tilde{\mathbb{P}}_{\iota_N, c}^N(\mathcal{I}^N(\varphi) \geq -\delta) = 1$ where*

$$\begin{aligned} \mathcal{I}^N(\varphi) := & \int_0^\infty \left(\left\langle \partial_s \varphi(s, \cdot), (\eta_s^N(\cdot) - \zeta_s^N(\cdot))^\pm \right\rangle_N + \left\langle \partial_x \varphi(s, \cdot), H^\pm(\tau, \eta_s^N, \tau, \zeta_s^N) \right\rangle_N \right. \\ & \left. + 2 \left((1-c)^\pm \varphi(s, 0) + (0-c)^\pm \varphi(s, 1) \right) \right) ds + \left\langle \varphi(0, \cdot), (\eta_0^N(\cdot) - \zeta_0^N(\cdot))^\pm \right\rangle_N, \end{aligned}$$

where $H^+(\eta, \zeta) = -2(b(\eta(1), \eta(0)) - b(\zeta(1), \zeta(0)))F_{1,0}(\eta, \zeta)$ and $H^-(\eta, \zeta) = H^+(\zeta, \eta)$.

This is an adaptation of Theorem 3.1 in [Rez91].

Proof. We define

$$\begin{aligned} B_t = & \int_0^t \left(\left\langle \partial_s \varphi(s, \cdot), (\eta_s(\cdot) - \zeta_s(\cdot))^\pm \right\rangle_N + \tilde{\mathcal{L}} \langle \varphi(s, \cdot), (\eta_s(\cdot) - \zeta_s(\cdot))^\pm \rangle_N \right) ds \\ & + \left\langle \varphi(0, \cdot), (\eta_0^N(\cdot) - \zeta_0^N(\cdot))^\pm \right\rangle_N. \end{aligned}$$

We have the identity

$$\left\langle \varphi(t, \cdot), (\eta_t^N(\cdot) - \zeta_t^N(\cdot))^\pm \right\rangle_N = B_t + M_t, \quad (4.27)$$

where M is a mean zero martingale. Since φ has compact support, the l.h.s. vanishes for t large enough. Below, we work at an arbitrary time s so we drop the subscript s in the calculations. Moreover, we write $\varphi(k)$ instead of $\varphi(k/2N)$ to simplify notations. We treat separately the boundary part and the bulk part of the generator. Regarding the former, we have

$$\begin{aligned} & \tilde{\mathcal{L}}^{\text{bdry}} \langle \varphi(\cdot), (\eta(\cdot) - \zeta(\cdot))^+ \rangle_N \\ &= (2N)^\alpha (2p_N - 1) \left(\varphi(1)\eta(1)\zeta(1)(1-c) - \varphi(2N)\eta(2N)(1-\zeta(2N))c \right) \\ &\leq 2\varphi(0)(1-c) + \mathcal{O}(N^{-\alpha}), \end{aligned}$$

since φ is non-negative and $2p_N - 1 \sim 2(2N)^{-\alpha}$. Similarly, we find

$$\tilde{\mathcal{L}}^{\text{bdry}} \langle \varphi(\cdot), (\eta(\cdot) - \zeta(\cdot))^- \rangle_N \leq 2\varphi(2N)(0-c)^- + \mathcal{O}(N^{-\alpha}).$$

We turn to the bulk part of the generator. Recall the map $F_{k,\ell}(\eta, \zeta)$, and set $G_{k,\ell}(\eta, \zeta) = 1 - F_{k,\ell}(\eta, \zeta)F_{k,\ell}(\zeta, \eta)$. By checking all the possible cases, one easily gets the following identity

$$\begin{aligned} \tilde{\mathcal{L}}^{\text{bulk}}(\eta(k) - \zeta(k))^+ = & (2N)^{1+\alpha} \sum_{\ell} \left[\left(p(\ell - k)(b(\zeta(k), \zeta(\ell)) - b(\eta(k), \eta(\ell))) \right. \right. \\ & \left. \left. - p(k - \ell)(b(\zeta(\ell), \zeta(k)) - b(\eta(\ell), \eta(k))) \right) F_{k,\ell}(\eta, \zeta) \right. \\ & \left. - \left(p(\ell - k)b(\eta(k), \eta(\ell)) + p(k - \ell)b(\zeta(\ell), \zeta(k)) \right) G_{k,\ell}(\eta, \zeta) \right]. \end{aligned}$$

Since η and ζ play symmetric rôles in $\tilde{\mathcal{L}}^{\text{bulk}}$, we find a similar identity for $\tilde{\mathcal{L}}^{\text{bulk}}(\eta(k) - \zeta(k))^-$. Notice that the term on the third line is non-positive, so we will drop it in the inequalities below. We thus get

$$\tilde{\mathcal{L}}^{\text{bulk}}\langle\varphi(\cdot), (\eta(\cdot) - \zeta(\cdot))^{\pm}\rangle_N \leq (2N)^{\alpha} \sum_{\substack{k, \ell=1 \\ \ell=k\pm 1}}^{2N} p(\ell - k)(\varphi(k) - \varphi(\ell))I_{k, \ell}^{\pm}(\eta, \zeta),$$

where

$$I_{k, \ell}^{+}(\eta, \zeta) = (b(\zeta(k), \zeta(\ell)) - b(\eta(k), \eta(\ell)))F_{k, \ell}(\eta, \zeta), \quad I_{k, \ell}^{-}(\eta, \zeta) = I_{k, \ell}^{+}(\zeta, \eta).$$

Up to now, we essentially followed the calculations made in the first step of the proof of [Rez91, Thm 3.1]. At this point, we argue differently: we decompose $p(\pm 1)$ into the symmetric part $1 - p_N$, which is of order $1/2$, and the asymmetric part which is either 0 or $2p_N - 1 \sim 2(2N)^{-\alpha}$.

We start with the contribution of the symmetric part. Recall the definition of the number of sign changes n and of the integers $k_1 < \dots < k_{n+1}$. Using a discrete integration by parts, one easily deduces that for all $i \leq n$

$$\begin{aligned} \sum_{\substack{k, \ell=k_i \\ \ell=k\pm 1}}^{k_{i+1}-1} (\varphi(k) - \varphi(\ell))I_{k, \ell}^{\pm}(\eta, \zeta) &= \sum_{k=k_i}^{k_{i+1}-2} (\eta(k) - \zeta(k))^{\pm} \Delta\varphi(k) \\ &\quad - (\eta(k_{i+1} - 1) - \zeta(k_{i+1} - 1))^{\pm} \nabla\varphi(k_{i+1} - 2) \\ &\quad + (\eta(k_i) - \zeta(k_i))^{\pm} \nabla\varphi(k_i - 1). \end{aligned}$$

Since $n(s)$ is bounded uniformly over all $N \geq 1$ and all $s \geq 0$, we deduce that the boundary terms arising at the second and third lines yield a negligible contribution. Thus we find

$$(2N)^{\alpha} \sum_{\substack{k, \ell=1 \\ \ell=k\pm 1}}^{2N} (1 - p_N)(\varphi(k) - \varphi(\ell))I_{k, \ell}^{\pm}(\eta, \zeta) = \mathcal{O}\left(\frac{1}{N^{1-\alpha}}\right).$$

Regarding the asymmetric part $p(\pm 1) - (1 - p_N)$, a simple calculation yields the identity

$$\begin{aligned} (2N)^{\alpha} \sum_{\substack{k, \ell=1 \\ \ell=k\pm 1}}^{2N} (p(\ell - k) - 1 + p_N)(\varphi(k) - \varphi(\ell))I_{k, \ell}^{\pm}(\eta, \zeta) \\ = \frac{1}{2N} \sum_{k=1}^{2N-1} \partial_x \varphi(k) \tau_k H^{\pm}(\eta, \zeta) + \mathcal{O}(N^{-\alpha}), \end{aligned}$$

uniformly over all $N \geq 1$. Therefore

$$\tilde{\mathcal{L}}^{\text{bulk}}\langle\varphi(\cdot), (\eta(\cdot) - \zeta(\cdot))^{\pm}\rangle_N \leq \frac{1}{2N} \sum_{k=1}^{2N-1} \partial_x \varphi(k) \tau_k H^{\pm}(\eta, \zeta) + \mathcal{O}\left(\frac{1}{N^{\alpha \wedge (1-\alpha)}}\right).$$

Putting together the two contributions of the generator, we get

$$\begin{aligned} B_t \leq & \int_0^t \left(\langle \partial_s \varphi(s, \cdot), (\eta_s^N(\cdot) - \zeta_s^N(\cdot))^{\pm} \rangle_N + \langle \partial_x \varphi(s, \cdot), \tau H^{\pm}(\eta_s^N, \zeta_s^N) \rangle_N \right. \\ & + 2t((1-c)^{\pm} \varphi(s, 0) + (0-c)^{\pm} \varphi(s, 1)) \Big) ds \\ & + \left\langle \varphi(0, \cdot), (\eta_0^N(\cdot) - \zeta_0^N(\cdot))^{\pm} \right\rangle_N + \mathcal{O}\left(\frac{1}{N^{\alpha \wedge (1-\alpha)}}\right). \end{aligned}$$

Recall the equation (4.27). A simple calculation shows that $\tilde{\mathbb{E}}_{\iota_N, c}^N \langle M \rangle_t \lesssim \frac{1}{N^{1-\alpha}}$ uniformly over all $N \geq 1$ and all $t \geq 0$. Moreover, the jumps of M are almost surely bounded by a term of order N^{-1} . Applying the BDG inequality (6.3), we deduce that

$$\tilde{\mathbb{E}}_{\iota_N, c}^N \left[\sup_{s \leq t} M_s^2 \right]^{\frac{1}{2}} \lesssim \frac{1}{N^{\frac{1-\alpha}{2}}},$$

uniformly over all $N \geq 1$ and all $t \geq 0$. Since φ has compact support, $B_t = -M_t$ for t large enough. The assertion of the lemma then easily follows. \square

Recall that $\mathcal{M}_{T_\ell(u)}\eta$ is the average of η on the box $T_\ell(u)$ for any $u \in \{1, \dots, 2N\}$.

Lemma 4.18 (Macroscopic inequalities) *Let ι_N be a measure on $\{0, 1\}^{2N}$ satisfying Assumption 4.1. For all $\varphi \in \mathcal{C}_c^\infty([0, \infty) \times [0, 1], \mathbf{R}_+)$, all $\delta > 0$ and all $c \in [0, 1]$, we have $\lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \mathbb{P}_{\iota_N}^N(\mathcal{J}^N(\varphi) \geq -\delta) = 1$ where*

$$\begin{aligned} \mathcal{J}^N(\varphi) := & \int_0^\infty \left(\left\langle \partial_s \varphi(s, \cdot), \left(\mathcal{M}_{T_{\epsilon N}(\cdot)}(\eta_s^N) - c \right)^{\pm} \right\rangle_N \right. \\ & + \left\langle \partial_x \varphi(s, \cdot), h^{\pm} \left(\mathcal{M}_{T_{\epsilon N}(\cdot)}(\eta_s^N), c \right) \right\rangle_N \\ & + 2 \left((1-c)^{\pm} \varphi(s, 0) + (0-c)^{\pm} \varphi(s, 1) \right) \Big) ds \\ & + \left\langle \varphi(0, \cdot), \left(\mathcal{M}_{T_{\epsilon N}(\cdot)}(\eta_0^N) - c \right)^{\pm} \right\rangle_N. \end{aligned} \quad (4.28)$$

Proof. Since at any time $s \geq 0$, $\zeta^N(s, \cdot)$ is distributed according to a product of Bernoulli measures with parameter c , we deduce that

$$\lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \tilde{\mathbb{E}}_{\iota_N, c}^N \left[\frac{1}{2N} \sum_{u=1}^{2N} \left| \mathcal{M}_{T_{\epsilon N}(u)}(\zeta_s^N) - c \right| \right] = 0.$$

and consequently, by Fubini's Theorem and stationarity, we have

$$\lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \tilde{\mathbb{E}}_{\iota_N, c}^N \left[\int_0^t \frac{1}{2N} \sum_{u=1}^{2N} \left| \mathcal{M}_{T_{\epsilon N}(u)}(\zeta_s^N) - c \right| ds \right] = 0.$$

Now we observe that for all $\epsilon > 0$, we have $\tilde{\mathbb{P}}_{\iota_N, c}^N$ almost surely

$$\left\langle \varphi(0, \cdot), (\eta_0^N(\cdot) - \zeta_0^N(\cdot))^{\pm} \right\rangle_N = \left\langle \varphi(0, \cdot), \mathcal{M}_{T_{\epsilon N}(\cdot)}(\eta_0^N - \zeta_0^N)^{\pm} \right\rangle_N + \mathcal{O}(\epsilon).$$

Recall the coupling we chose for $(\eta_0^N(\cdot), \zeta_0^N(\cdot))$. Since $\tilde{\mathbb{P}}_{\iota_N, c}^N$ almost surely the number of sign changes $n(0)$ is bounded by some constant $C > 0$ uniformly over all $N \geq 1$, we deduce using the previous identity that

$$\left\langle \varphi(0, \cdot), (\eta_0^N(\cdot) - \zeta_0^N(\cdot))^{\pm} \right\rangle_N = \left\langle \varphi(0, \cdot), (\mathcal{M}_{T_{\epsilon N}(\cdot)} \eta_0^N - \mathcal{M}_{T_{\epsilon N}(\cdot)} \zeta_0^N)^{\pm} \right\rangle_N + \mathcal{O}(\epsilon).$$

Therefore, by Lemma 4.17, we deduce that the statement of the lemma follows if we can show that for all $\delta > 0$

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \tilde{\mathbb{P}}_{\iota_N, c}^N \left(\int_0^t \frac{1}{2N} \sum_{u=1}^{2N} \left| \mathcal{M}_{T_{\epsilon N}(u)} (\eta_s^N - \zeta_s^N)^{\pm} \right. \right. \\ \left. \left. - (\mathcal{M}_{T_{\epsilon N}(u)} (\eta_s^N - \zeta_s^N))^{\pm} \right| ds > \delta \right) = 0, \\ \lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \tilde{\mathbb{P}}_{\iota_N, c}^N \left(\int_0^t \frac{1}{2N} \sum_{u=1}^{2N} \left| \mathcal{M}_{T_{\epsilon N}(u)} H^{\pm}(\eta_s^N, \zeta_s^N) \right. \right. \\ \left. \left. - h^{\pm}(\mathcal{M}_{T_{\epsilon N}(u)}(\eta_s^N), \mathcal{M}_{T_{\epsilon N}(u)}(\zeta_s^N)) \right| ds > \delta \right) = 0. \end{aligned} \quad (4.29)$$

We restrict ourselves to proving the second identity, since the first is simpler. Let \mathcal{N}_s^+ , resp. \mathcal{N}_s^- , be the set of $u \in \{1, \dots, 2N\}$ such that $\eta_s \geq \zeta_s$, resp. $\zeta_s \geq \eta_s$, on the whole box $T_{\epsilon N}(u)$. By Lemma 4.16, $2N - \#\mathcal{N}_s^+ - \#\mathcal{N}_s^-$ is of order ϵN uniformly over all s , all $N \geq 1$ and all ϵ . Therefore, we can neglect the contribution of all $u \notin \mathcal{N}_s^+ \cup \mathcal{N}_s^-$. If we define $\Phi(\eta) = -2\eta(1)(1 - \eta(0))$ and if we let $\tilde{\Phi}(a)$ be as in (4.1), then for all $u \in \mathcal{N}_s^+$ we have

$$\mathcal{M}_{T_{\epsilon N}(u)} H^-(\eta_s^N, \zeta_s^N) - h^-\left(\mathcal{M}_{T_{\epsilon N}(u)}(\eta_s^N), \mathcal{M}_{T_{\epsilon N}(u)}(\zeta_s^N)\right) = 0,$$

as well as

$$\begin{aligned} & \mathcal{M}_{T_{\epsilon N}(u)} H^+(\eta_s^N, \zeta_s^N) - h^+\left(\mathcal{M}_{T_{\epsilon N}(u)}(\eta_s^N), \mathcal{M}_{T_{\epsilon N}(u)}(\zeta_s^N)\right) \\ &= \mathcal{M}_{T_{\epsilon N}(u)} \Phi(\eta_s^N) - \tilde{\Phi}\left(\mathcal{M}_{T_{\epsilon N}(u)} \eta_s^N\right) - \mathcal{M}_{T_{\epsilon N}(u)} \Phi(\zeta_s^N) + \tilde{\Phi}\left(\mathcal{M}_{T_{\epsilon N}(u)} \zeta_s^N\right). \end{aligned}$$

Similar identities hold for every $u \in \mathcal{N}_s^-$. We deduce that (4.29) follows if we can show that for all $\delta > 0$

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \mathbb{P}_{\iota_N}^N \left(\int_0^t \frac{1}{2N} \sum_{u=1}^{2N} \left| \mathcal{M}_{T_{\epsilon N}(u)} \Phi(\eta_s^N) - \tilde{\Phi}\left(\mathcal{M}_{T_{\epsilon N}(u)} \eta_s^N\right) \right| ds > \delta \right) = 0, \\ \lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \tilde{\mathbb{E}}_{\iota_N, c}^N \left[\int_0^t \frac{1}{2N} \sum_{u=1}^{2N} \left| \mathcal{M}_{T_{\epsilon N}(u)} \Phi(\zeta_s^N) - \tilde{\Phi}\left(\mathcal{M}_{T_{\epsilon N}(u)} \zeta_s^N\right) \right| ds \right] = 0. \end{aligned}$$

The first convergence is ensured by Theorem 4.2, while the second follows from the stationarity of ζ^N and the Ergodic Theorem. This completes the proof of the lemma. \square

Proof of Theorem 4.14. For any given $\epsilon > 0$, we have

$$\begin{aligned} \mathcal{M}_{T_{2\epsilon N}(k)}(\eta_s) &= \frac{1}{2\epsilon} \varrho^N \left(s, \left[\frac{k}{2N} - \epsilon, \frac{k}{2N} + \epsilon \right] \right) \\ &= \frac{1}{2\epsilon} \varrho^N \left(s, [x - \epsilon, x + \epsilon] \right) + \mathcal{O}(N^{-1}), \end{aligned} \quad (4.30)$$

uniformly over all $k \in \{1, \dots, 2N-1\}$, all $x \in \left[\frac{k}{2N}, \frac{k+1}{2N}\right]$ and all $N \geq 1$. Notice that the $\mathcal{O}(N^{-1})$ depends on ϵ . For all $\varrho \in \mathbb{D}([0, \infty), \mathcal{M}([0, 1]))$, we set

$$\begin{aligned} V_c(\epsilon, \varrho) := & \int_0^\infty \left\langle \partial_s \varphi(s, \cdot), \left(\frac{1}{2\epsilon} \varrho \left(s, \left[\cdot - \epsilon, \cdot + \epsilon \right] \right) - c \right)^\pm \right\rangle \\ & + \left\langle \partial_x \varphi(s, \cdot), h^\pm \left(\frac{1}{2\epsilon} \varrho \left(s, \left[\cdot - \epsilon, \cdot + \epsilon \right] \right), c \right) \right\rangle \\ & + 2 \left((1-c)^\pm \varphi(s, 0) + (0-c)^\pm \varphi(s, 1) \right) ds \\ & + \left\langle \varphi(0, \cdot), \left(\frac{1}{2\epsilon} \varrho \left(0, \left[\cdot - \epsilon, \cdot + \epsilon \right] \right) - c \right)^\pm \right\rangle. \end{aligned}$$

Combining (4.30), (4.28) and the continuity of the maps $h^\pm(\cdot, c)$ and $(\cdot)^\pm$, we deduce that for any $\delta > 0$, we have

$$\lim_{\epsilon \downarrow 0} \lim_{N \rightarrow \infty} \mathbb{P}_{\iota_N}^N (V_c(\epsilon, \varrho^N) \geq -\delta) = 1.$$

At this point, we observe that for all $\varphi \in \mathcal{C}([0, 1], \mathbf{R}_+)$ we have

$$\langle \varrho^N(t), \varphi \rangle \leq \frac{1}{2N} \sum_{k=1}^{2N} \varphi(k/2N),$$

so that a simple argument ensures that for every limit point ϱ of ϱ^N and for all $t \geq 0$, the measure $\varrho(t, dx)$ is absolutely continuous with respect to the Lebesgue measure, and its density is bounded by 1. Therefore, any limit point is of the form $\varrho(t, dx) = \eta(t, x)dx$ with $\eta \in L^\infty([0, \infty) \times (0, 1))$. Let \mathbb{P} be the law of the limit of a converging subsequence ϱ^{N_i} . Since $\varrho \mapsto V_c(\epsilon, \varrho)$ is a \mathbb{P} -a.s. continuous map on $\mathbb{D}([0, \infty), \mathcal{M}([0, 1]))$, we have for all $\epsilon > 0$

$$\overline{\lim}_{i \rightarrow \infty} \mathbb{P}_{\iota_{N_i}}^{N_i} (V_c(\epsilon, \varrho^{N_i}) \geq -\delta) \leq \mathbb{P}(V_c(\epsilon, \varrho) \geq -\delta).$$

For any ϱ of the form $\varrho(t, dx) = \eta(t, x)dx$, we set

$$\begin{aligned} V_c(\varrho) := & \int_0^\infty \left\langle \partial_s \varphi(s, \cdot), \left(\eta(s, \cdot) - c \right)^\pm \right\rangle + \left\langle \partial_x \varphi(s, \cdot), h^\pm \left(\eta(s, \cdot), c \right) \right\rangle \\ & + 2 \left((1-c)^\pm \varphi(s, 0) + (0-c)^\pm \varphi(s, 1) \right) ds + \left\langle \varphi(0, \cdot), (\eta_0 - c)^\pm \right\rangle, \end{aligned}$$

and we observe that by Lebesgue Differentiation Theorem, we have \mathbb{P} -a.s. $V_c(\varrho) = \lim_{\epsilon \downarrow 0} V_c(\epsilon, \varrho)$. Therefore,

$$\begin{aligned} \mathbb{P}(V_c(\varrho) \geq -\delta) &= \mathbb{P}(\lim_{\epsilon \downarrow 0} V_c(\epsilon, \varrho) \geq -\delta) \geq \mathbb{E}[\overline{\lim}_{\epsilon \downarrow 0} \mathbf{1}_{\{V_c(\epsilon, \varrho) \geq -\delta/2\}}] \\ &\geq \overline{\lim}_{\epsilon \downarrow 0} \mathbb{E}[\mathbf{1}_{\{V_c(\epsilon, \varrho) \geq -\delta/2\}}] \geq \overline{\lim}_{\epsilon \downarrow 0} \overline{\lim}_{i \rightarrow \infty} \mathbb{P}_{\iota_{N_i}}^{N_i} (V_c(\epsilon, \varrho^{N_i}) \geq -\delta/2) = 1, \end{aligned}$$

so the process $(\eta(t, x), t \geq 0, x \in (0, 1))$ under \mathbb{P} coincides with the unique entropy solution of (1.11), thus concluding the proof. \square

5 KPZ fluctuations*

To prove Theorem 1.8, we follow the method of Bertini and Giacomin [BG97]. Due to our boundary conditions, there are two important steps that need some specific arguments: first the bound on the moments of the discrete process, see Proposition 5.3, second the bound on the error terms arising in the identification of the limit, see Proposition 5.7.

In order to simplify the notations, we will regularly use the microscopic variables $k, \ell \in \{1, \dots, 2N - 1\}$ in rescaled quantities: for instance, $h^N(t, \ell)$ stands for $h^N(t, x)$ with $x = (\ell - N)/(2N)^{2\alpha}$. The proof relies on the discrete Hopf-Cole transform, which was introduced by Gärtner [Gär88]. The important feature of this transform is that it linearises the drift terms of the stochastic differential equations solved by the discrete process. Indeed, if one sets $\xi^N(t, x) := \exp(-h^N(t, x))$, where h^N was introduced in (1.15), then the stochastic differential equations solved by ξ^N are given by

$$\begin{cases} d\xi^N(t, \ell) = c_N \Delta \xi^N(t, \ell) dt + dM^N(t, \ell), \\ \xi^N(t, 0) = \xi^N(t, 2N) = e^{\lambda_N t}, \\ \xi^N(0, \ell) = e^{-h^N(0, \ell)}, \end{cases} \quad (5.1)$$

for all $\ell \in \{1, \dots, 2N - 1\}$, where M^N is a martingale with bracket given by $\langle M^N(\cdot, k), M^N(\cdot, \ell) \rangle_t = 0$ whenever $k \neq \ell$, and

$$\begin{aligned} d\langle M^N(\cdot, k) \rangle_t &= \lambda_N \left(\xi^N(t, k) \Delta \xi^N(t, k) + 2\xi^N(t, k)^2 \right) dt \\ &\quad - (2N)^{4\alpha} \nabla^+ \xi^N(t, k) \nabla^- \xi^N(t, k) dt, \end{aligned} \quad (5.2)$$

where we rely on the notation

$$\nabla^+ f(\ell) := f(\ell + 1) - f(\ell), \quad \nabla^- f(\ell) := f(\ell) - f(\ell - 1).$$

Observe that

$$|d\langle M^N(\cdot, k) \rangle_t| \lesssim \xi^N(t, k)^2 (2N)^{2\alpha},$$

uniformly over all $t \geq 0$, all k and all $N \geq 1$. As usual, we let $\mathcal{F}_t, t \geq 0$ be the natural filtration associated with the process $(\xi^N(t), t \geq 0)$. In order to analyse this random process, we need to define a few objects first.

We define $B_0^N(t) := [\frac{\lambda_N}{\gamma_N} t, 2N - \frac{\lambda_N}{\gamma_N} t] \subset [0, 2N]$. The hydrodynamic limit obtained in Theorem 1.5 shows that this is the window where the density of particles is approximately $1/2$ at time t (in the time scale $(2N)^{4\alpha}$). On the left of this window, the density is approximately 1, and on the right it is approximately 0. For technical reasons, it is convenient to introduce an ϵ -approximation of this window by setting:

$$B_\epsilon^N(t) := \left[\frac{\lambda_N}{\gamma_N} t + \epsilon N, 2N - \frac{\lambda_N}{\gamma_N} t - \epsilon N \right], \quad t \in [0, T].$$

We let $p_t^N(k, \ell)$ be the discrete heat kernel on $\{0, \dots, 2N\}$ sped up by $2c_N$, we refer to Appendix 6.2 for a definition and some properties. Classical arguments ensure that the unique solution of (5.1) is given by

$$\xi^N(t, \ell) = I^N(t, \ell) + N_t^t(\ell), \quad (5.3)$$

*This section is taken from [Lab16]

where, for all $t \geq 0$, $[0, t] \ni r \mapsto N_r^t(\ell)$ is the martingale

$$N_r^t(\ell) = \int_0^r \sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell) dM^N(s, k), \quad (5.4)$$

and $I^N(t, \ell)$ is the term coming from the initial condition.

We define

$$b^N(t, \ell) := 2 + \exp\left(\lambda_N t - \gamma_N(\ell \wedge (2N - \ell))\right).$$

Remark 5.1 The hydrodynamic limit of Theorem 1.5, upon Hopf-Cole transform, is given by $1 \vee \exp\left(\lambda_N t - \gamma_N(\ell \wedge (2N - \ell))\right)$.

Proposition 5.2 *Let K be a compact subset of $[0, T]$ and fix $\epsilon > 0$. Uniformly over all $t \in K$, we have*

- $I^N(t, \ell) \lesssim b^N(t, \ell)$ for all $\ell \in \{1, \dots, 2N\}$,
- $|\nabla^\pm I^N(t, \ell)| \lesssim t^{-\frac{1}{2}} N^{-3\alpha}$ uniformly over all $\ell \in B_\epsilon^N(t)$,
- $|I^N(t, \ell) - I^N(t', \ell)| \lesssim N^{-\alpha}$ uniformly over all $\ell \in B_\epsilon^N(t')$ and all $t < t' \in K$,
- $|I^N(t, \ell) - I^N(t, \ell')| \lesssim N^{-\alpha}$ uniformly over all $\ell, \ell' \in B_\epsilon^N(t)$.

Proof. We write

$$I^N(t, \ell) = \xi^{N, \circ}(t, \ell) + \sum_{k=1}^{2N-1} p_t^N(k, \ell)(\xi^N(0, k) - 1),$$

where $\xi^{N, \circ}$ is the solution of

$$\begin{cases} \partial_t \xi^{N, \circ}(t, \ell) = c_N \Delta \xi^{N, \circ}(t, \ell), \\ \xi^{N, \circ}(t, 0) = \xi^{N, \circ}(t, 2N) = e^{\lambda_N t}, \\ \xi^{N, \circ}(0, \ell) = 1. \end{cases}$$

Since our initial condition is flat, it is immediate to check that

$$\left| \sum_{k=1}^{2N-1} p_t^N(k, \ell)(\xi^N(0, k) - 1) \right| \lesssim N^{-\alpha} \ll b^N(t, \ell),$$

which immediately yield the first, third and fourth bounds of the statement for this term. Furthermore, using Lemmas 6.4 and 6.5, we get

$$\nabla^\pm(I^N(t, \ell) - \xi^{N, \circ}(t, \ell)) = \sum_{k \in B_{\epsilon/2}^N(0)} \nabla^+ \bar{p}_t^N(\ell - k)(\xi^N(0, k) - 1) + \mathcal{O}(N^{1-\alpha} e^{-\delta N^{2\alpha}}),$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $t \in K$ and all $N \geq 1$. Then, we write

$$\sum_{k \in B_{\epsilon/2}^N(0)} |\nabla^+ \bar{p}_t^N(\ell - k)| = -\bar{p}_t^N(\ell - i_- - 1) + 2\bar{p}_t^N(0) - \bar{p}_t^N(\ell - i_+),$$

where i_{\pm} are the first and last integers in $B_{\epsilon/2}^N(0)$. Using Lemma 6.2 and our choice of initial condition, we deduce that

$$\left| \sum_{k \in B_{\epsilon/2}^N(0)} \nabla^+ \bar{p}_t^N(\ell - k)(\xi^N(0, k) - 1) \right| \lesssim 1 \wedge \frac{1}{\sqrt{t}(2N)^{3\alpha}},$$

uniformly over the same set of parameters. The same applies to ∇^- .

To establish the required bounds on $\xi^{N,\circ}$, we first show that there exists $\delta > 0$ such that

$$|\xi^{N,\circ}(t, \ell) - 1| \lesssim \exp(-\delta N^{2\alpha}), \quad (5.5)$$

uniformly over all $t \in K$, all $\ell \in B_{\epsilon}^N(t)$ and all $N \geq 1$. Since

$$\xi^{N,\circ}(t, \ell) = 1 + \lambda_N \int_0^t \left(1 - \sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell) \right) e^{\lambda_N s} ds,$$

the bound will be ensured if we are able to show that there exists $\delta > 0$ such that

$$\left(1 - \sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell) \right) e^{\lambda_N s} \lesssim e^{-\delta N^{2\alpha}}, \quad (5.6)$$

uniformly over all $s \in [0, t]$, all $t \in K$ and all $\ell \in B_{\epsilon}^N(t)$. The proof of this estimate on the heat kernel is provided in Appendix 6.2. This yields (5.5), and therefore concludes the proof of the second, third and fourth bounds of the statement. Using the estimate on $\xi^{N,\circ}(t, N) - 1$ obtained above, we deduce that for N large enough, b^N solves

$$\begin{cases} \partial_t b^N(t, \ell) = c_N \Delta b^N(t, \ell), & \ell \in \{1, \dots, N-1\}, \\ b^N(t, 0) \geq \xi^{N,\circ}(t, 0), & b^N(t, N) \geq \xi^{N,\circ}(t, N), \\ b^N(0, k) \geq \xi^{N,\circ}(0, k). \end{cases}$$

By the maximum principle, one deduces that $b^N(t, \ell) \geq \xi^{N,\circ}(t, \ell)$ for all $t \in K$ and all $\ell \in \{0, \dots, N\}$. By symmetry, this inequality also holds for $\ell \in \{N, \dots, 2N\}$. \square

To alleviate the notation, we define

$$q_{s,t}^N(k, \ell) = p_{t-s}^N(k, \ell) b^N(s, k). \quad (5.7)$$

We now have all the ingredients at hand to bound the moments of ξ^N .

Proposition 5.3 *For all $n \geq 1$ and all compact set $K \subset [0, T]$, we have*

$$\sup_{N \geq 1} \sup_{\ell \in \{1, \dots, 2N-1\}} \sup_{t \in K} \mathbb{E} \left[\left(\frac{\xi^N(t, \ell)}{b^N(t, \ell)} \right)^n \right] < \infty.$$

Proof. We fix the compact set K until the end of the proof. Using the expression (5.3) and Proposition 5.2, we deduce that

$$\mathbb{E} \left[\left(\frac{\xi^N(t, \ell)}{b^N(t, \ell)} \right)^{2n} \right]^{\frac{1}{2n}} \leq 1 + \mathbb{E} \left[\left(\frac{N_t^t(\ell)}{b^N(t, \ell)} \right)^{2n} \right]^{\frac{1}{2n}}. \quad (5.8)$$

We set $D_r^t := [N^t]_r - \langle N^t \rangle_r$ and we refer to Appendix 6.1 for the notations. By the BDG inequality (6.2), we obtain

$$\mathbb{E} \left[(N_t^t(\ell))^{2n} \right] \lesssim \mathbb{E} \left[\langle N^t(\ell) \rangle_t^n \right] + \mathbb{E} \left[[D^t(\ell)]_t^{\frac{n}{2}} \right], \quad (5.9)$$

uniformly over all $\ell \in \{1, \dots, 2N-1\}$, all $t \geq 0$, and all $N \geq 1$. Let

$$g_n^N(s) := \sup_{k \in \{1, \dots, 2N-1\}} \mathbb{E} \left[\left(\frac{\xi^N(s, k)}{b^N(s, k)} \right)^{2n} \right].$$

We claim that

$$\mathbb{E} \left[\langle N^t(\ell) \rangle_t^n \right] \lesssim b^N(t, \ell)^{2n} \int_0^t \frac{g_n^N(s)}{\sqrt{t-s}} ds, \quad (5.10)$$

$$\mathbb{E} \left[[D^t(\ell)]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim b^N(t, \ell)^4 \left(1 + \int_0^t \frac{g_n^N(s)^{\frac{2}{n}}}{\sqrt{t-s} (2N)^{2\alpha}} ds \right), \quad (5.11)$$

uniformly over all $\ell \in \{1, \dots, 2N-1\}$, all $N \geq 1$ and all $t \in K$. We postpone the proof of these two bounds. Combining these two bounds with (5.8) and (5.9), we obtain the following closed inequality

$$g_n^N(t) \lesssim 1 + \int_0^t \frac{g_n^N(s)}{\sqrt{t-s}} ds,$$

uniformly over all $N \geq 1$ and all $t \in K$. By a generalised Grönwall's inequality, see for instance [Har81, Lemma 6 p.33], we deduce that g_n^N is uniformly bounded over all $N \geq 1$ and all $t \in K$.

We are left with establishing (5.10) and (5.11). Using (5.2), we obtain the almost sure bound

$$\langle N^t(\ell) \rangle_t \lesssim (2N)^{2\alpha} \int_0^t \sum_k p_{t-s}^N(k, \ell)^2 \xi^N(s, k)^2 ds,$$

uniformly over all $N \geq 1$, $t \geq 0$ and $\ell \in \{1, \dots, 2N-1\}$. Recall the function q^N from (5.7). Using Hölder's inequality at the first line, Lemma 6.3 at the second and Jensen's inequality at the third, we get

$$\begin{aligned} \mathbb{E} \left[\left(\frac{\langle N^t(\ell) \rangle_t}{b^N(t, \ell)^2} \right)^n \right] &\lesssim \int_{s_1, \dots, s_n=0}^t \sum_{k_1, \dots, k_n} \prod_{i=1}^n (2N)^{2\alpha} \left(\frac{q_{s_i, t}^N(k_i, \ell)}{b^N(t, \ell)} \right)^2 g_n(s_i)^{\frac{1}{n}} ds_i \\ &\lesssim \left(\int_0^t \frac{g_n(s)^{\frac{1}{n}}}{\sqrt{t-s}} ds \right)^n \\ &\lesssim \int_0^t \frac{g_n(s)}{\sqrt{t-s}} ds, \end{aligned}$$

uniformly over all $N \geq 1$, all $t \in K$ and all $\ell \in \{1, \dots, 2N-1\}$, thus yielding (5.10).

We turn to the quadratic variation. Let J_k be the set of jump times of $\xi^N(\cdot, k)$. We start with the following simple bound

$$[D^t(\ell)]_t = \sum_{\tau \leq t} \sum_k p_{t-\tau}^N(k, \ell)^4 (\xi^N(\tau, k) - \xi^N(\tau-, k))^4$$

$$\lesssim \gamma_N^4 \sum_k \sum_{\tau \leq t; \tau \in J_k} q_{\tau,t}^N(k, \ell)^4 \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4,$$

uniformly over all $N \geq 1$, all $t \geq 0$ and all $\ell \in \{1, \dots, 2N-1\}$. We set $t_i := i(2N)^{-4\alpha}$ for all $i \in \mathbf{N}$ and we let $\mathcal{I}_i := [t_i, t_{i+1})$. Then, by Minkowski's inequality we have

$$\mathbb{E} \left[[D^t(\ell)]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim \gamma_N^4 \sum_{i=0}^{\lfloor t(2N)^{4\alpha} \rfloor} \sum_k \sup_{s \in \mathcal{I}_i, s < t} q_{s,t}^N(k, \ell)^4 \mathbb{E} \left[\left(\sum_{\tau \in \mathcal{I}_i \cap J_k} \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4 \right)^{\frac{n}{2}} \right]^{\frac{2}{n}},$$

Let $Q(k, r, s)$ be the number of jumps of the process $\xi^N(\cdot, k)$ on the time interval $[r, s]$. We have the following almost sure bound

$$\xi^N(\tau, k) \leq \xi^N(s, k) e^{2(2N)^{-4\alpha} \lambda_N + 2\gamma_N Q(k, s, t_{i+1})},$$

uniformly over all $s \in \mathcal{I}_{i-1}$, all $\tau \in \mathcal{I}_i$, all $k \in \{1, \dots, 2N-1\}$ and all $i \geq 1$. Consequently we get

$$\sum_{\tau \in \mathcal{I}_i \cap J_k} \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4 \lesssim (2N)^{4\alpha} \int_{t_{i-1}}^{t_i} \left(\frac{\xi^N(s, k)}{b^N(s, k)} \right)^4 Q(k, s, t_{i+1}) e^{8\gamma_N Q(k, s, t_{i+1})} ds,$$

uniformly over all $N \geq 1$, all $i \geq 1$ and all $k \in \{1, \dots, 2N-1\}$. Since $(Q(k, s, t), t \geq s)$ is, conditionally given \mathcal{F}_s , stochastically bounded by a Poisson process with rate $(2N)^{4\alpha}$, we deduce that there exists $C > 0$ such that almost surely

$$\sup_{N \geq 1} \sup_{i \geq 1} \sup_{s \in \mathcal{I}_{i-1}} \mathbb{E} \left[Q(k, s, t_{i+1})^{\frac{n}{2}} e^{4n\gamma_N Q(k, s, t_{i+1})} \middle| \mathcal{F}_s \right] < C.$$

Then, we get

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{\tau \in \mathcal{I}_i \cap J_k} \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4 \right)^{\frac{n}{2}} \right]^{\frac{2}{n}} \\ & \lesssim (2N)^{4\alpha} \int_{t_{i-1}}^{t_i} \mathbb{E} \left[\left(\left(\frac{\xi^N(s, k)}{b^N(s, k)} \right)^4 Q(k, s, t_{i+1}) e^{8\gamma_N Q(k, s, t_{i+1})} \right)^{\frac{n}{2}} \right]^{\frac{2}{n}} ds \\ & \lesssim C(2N)^{4\alpha} \int_{t_{i-1}}^{t_i} g_n(s)^{\frac{2}{n}} ds, \end{aligned}$$

uniformly over all $N \geq 1$, all $i \geq 1$ and all k . On the other hand, when $i = 0$ we have the following bound

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{\tau \in \mathcal{I}_0 \cap J_k} \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4 \right)^{\frac{n}{2}} \right]^{\frac{2}{n}} & \lesssim \left(\frac{\xi^N(0, k)}{b^N(0, k)} \right)^4 \mathbb{E} \left[Q(k, 0, t_1)^{\frac{n}{2}} e^{2n\gamma_N Q(k, 0, t_1)} \right] \\ & \lesssim 1, \end{aligned}$$

uniformly over all k and all $N \geq 1$.

Observe that

$$p_{t-s}^N(k, \ell) = e^{-2c_N(t-s)} \sum_{n \geq 0} \frac{(2c_N(t-s))^n}{n!} \mathbf{p}_n(k, \ell),$$

where $\mathbf{p}_n(k, \ell)$ is the probability that a discrete-time random walk, killed upon hitting 0 and $2N$ and started from k , reaches ℓ after n steps. Therefore, we easily deduce that $\sup_{s \in \mathcal{I}_i} q_{s,t}^N(\ell, k) \lesssim q_{t_i,t}^N(\ell, k)$. Using Lemma 6.3, we get

$$\sum_k \sup_{s \in \mathcal{I}_i, s < t} q_{s,t}^N(k, \ell)^4 \lesssim \sum_k q_{t_i,t}^N(k, \ell)^4 \lesssim b^N(t, \ell)^4 \left(1 \wedge \frac{1}{\sqrt{t-t_i} (2N)^{2\alpha}} \right),$$

uniformly over all $N \geq 1$ and $i \geq 0$. Putting everything together, we obtain

$$\mathbb{E} \left[[D^t(k)]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim b^N(t, \ell)^4 \left(1 + \int_0^t \frac{g_n(s)^{\frac{2}{n}}}{\sqrt{t-s} (2N)^{2\alpha}} ds \right),$$

as required, thus concluding the proof. \square

5.1 Tightness

We now establish some estimates on the moments of time and space increments. The following two lemmas are similar to Lemmas 4.2 and 4.3 in [BG97].

Lemma 5.4 *Fix $\epsilon > 0$, $\beta \in (0, 1/2)$ and a compact set $K \subset [0, T)$. For any $n \geq 1$, we have*

$$\mathbb{E} \left[|\xi^N(t, \ell') - \xi^N(t, \ell)|^{2n} \right]^{\frac{1}{2n}} \lesssim \left| \frac{\ell - \ell'}{(2N)^{2\alpha}} \right|^\beta,$$

uniformly over all $t \in K$, all $\ell, \ell' \in B_\epsilon^N(t)$ and all $N \geq 1$.

Proof. The expression (5.3) yields two terms for $\xi^N(t, \ell) - \xi^N(t, \ell')$. By Proposition 5.2, the first term can be bounded by a term of order $N^{-\alpha}$ which is negligible compared to $(|\ell - \ell'|/(2N)^{2\alpha})^\beta$ whenever $\ell \neq \ell'$. Therefore, to complete the proof of the lemma, we only need to establish the appropriate bound for the $2n$ -th moment $R_t^t(\ell, \ell')$, where we have introduced the martingale

$$R_s^t(\ell, \ell') := \int_0^r \sum_{k=1}^{2N-1} (p_{t-s}^N(k, \ell) - p_{t-s}^N(k, \ell')) dM^N(s, k), \quad r \in [0, t].$$

Let $D_s^t(\ell, \ell') = [R_s^t(\ell, \ell')]_s - \langle R_s^t(\ell, \ell') \rangle_s$. We claim that we have

$$\mathbb{E} \left[\langle R_s^t(\ell, \ell') \rangle_t^n \right]^{\frac{1}{n}} \lesssim \left| \frac{\ell - \ell'}{(2N)^{2\alpha}} \right|^{2\beta}, \quad \mathbb{E} \left[[D_s^t(\ell, \ell')]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim (2N)^{-4\alpha}, \quad (5.12)$$

uniformly over all $t \in K$, all $\ell, \ell' \in B_\epsilon^N(t)$ and all $N \geq 1$. These two inequalities, together with the BDG inequality (6.2) yield the desired bound on the $2n$ -th moment of $R_s^t(\ell, \ell')$, thus concluding the proof. We are left with the proof of these inequalities. As in the proof of Proposition 5.3, we observe that

$$\begin{aligned} \mathbb{E} \left[[D_s^t(\ell, \ell')]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} &\lesssim \gamma_N^4 \sum_{i=0}^{\lfloor t(2N)^{4\alpha} \rfloor} \sum_k \sup_{s \in \mathcal{I}_i, s < t} (q_{s,t}^N(k, \ell) - q_{s,t}^N(k, \ell'))^4 \\ &\quad \times \mathbb{E} \left[\left(\sum_{\tau \in \mathcal{I}_i} \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4 \right)^{\frac{n}{2}} \right]^{\frac{2}{n}}. \end{aligned}$$

The arguments in that proof ensure that the expectation in the r.h.s. is uniformly bounded over all i , all k and all $N \geq 1$. On the other hand, $\sup_{s \in \mathcal{I}_i} (q_{s,t}^N(k, \ell) - q_{s,t}^N(k, \ell'))^4 \lesssim q_{t_i,t}^N(k, \ell)^4 + q_{t_i,t}^N(k, \ell')^4$, so that Lemma 6.3 immediately yields

$$\sum_k \sup_{s \in \mathcal{I}_i} (q_{s,t}^N(k, \ell) - q_{s,t}^N(k, \ell'))^4 \lesssim 1 \wedge \left(\frac{1}{\sqrt{t - t_i} (2N)^{2\alpha}} \right)^3,$$

since $b^N(t, \ell)$ is of order 1 in $B_\epsilon^N(t)$. Hence, we get

$$\mathbb{E} \left[[D^t(\ell, \ell')]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim \gamma_N^4 \sum_{i=0}^{\lfloor t(2N)^{4\alpha} \rfloor} 1 \wedge \left(\frac{1}{\sqrt{t - t_i} (2N)^{2\alpha}} \right)^3 \lesssim \gamma_N^4,$$

uniformly over all $t \in K$, all $\ell, \ell' \in B_\epsilon^N(t)$ and all $N \geq 1$. This yields the second bound of (5.12). Regarding the first bound, we notice that we only have to consider the cases where $\ell \neq \ell'$. Then, we have the following almost sure bound

$$\langle R^t(\ell, \ell') \rangle_t \lesssim \int_0^t \sum_{k=1}^{2N-1} (p_{t-s}^N(k, \ell) - p_{t-s}^N(k, \ell'))^2 (2N)^{2\alpha} \xi^N(s, k)^2 ds.$$

We argue differently according as k belongs to $B_{\epsilon/2}^N(s)$ or not. Using Lemma 6.4, we deduce that

$$\int_0^t \sum_{k \notin B_{\epsilon/2}^N(s)} (q_{s,t}^N(k, \ell) - q_{s,t}^N(k, \ell'))^2 (2N)^{2\alpha} \mathbb{E} \left[\left(\frac{\xi^N(s, k)}{b^N(s, k)} \right)^{2n} \right]^{\frac{1}{n}} ds \lesssim N^{1+2\alpha} e^{-\delta N^{2\alpha}},$$

uniformly over all $\ell \in B_\epsilon(t)$, all $t \in K$ and all $N \geq 1$. This yields a bound of the desired order whenever $\ell \neq \ell'$. On the other hand, using Lemma 6.1 the contribution of the remaining k 's can be bounded as follows

$$\begin{aligned} & \int_0^t \sum_{k \in B_{\epsilon/2}^N(s)} (p_{t-s}^N(k, \ell) - p_{t-s}^N(k, \ell'))^2 (2N)^{2\alpha} \mathbb{E} [\xi^N(s, k)^{2n}]^{\frac{1}{n}} ds \\ & \lesssim \int_0^t \frac{ds}{(t-s)^{\frac{1}{2}}} \left| \frac{\ell - \ell'}{(2N)^{2\alpha}} \right|^{2\beta}, \end{aligned}$$

since $b^N(s, k)$ is of order 1 in $B_{\epsilon/2}^N(s)$, thus concluding the proof. \square

Lemma 5.5 Fix $\epsilon > 0$, $\beta \in (0, 1/4)$ and a compact set $K \subset [0, T]$. For any $n \geq 1$, we have

$$\mathbb{E} \left[|\xi^N(t', \ell) - \xi^N(t, \ell)|^{2n} \right]^{\frac{1}{2n}} \lesssim |t' - t|^\beta + \frac{1}{(2N)^\alpha},$$

uniformly over all $N \geq 1$, all $t < t' \in K$ and all $\ell \in B_\epsilon^N(t')$.

Proof. Using (5.3), we can write $\xi^N(t', \ell) - \xi^N(t, \ell)$ as the sum of two terms. Proposition 5.2 ensures that the first term is bounded by a term of order $N^{-\alpha}$ as required. Therefore, we only need to find the appropriate bound for the $2n$ -th moment of $N_{t'}^{t'}(\ell) - N_t^t(\ell)$. To that end, we bound separately the $2n$ -th moments of $A_\delta^{t, t'}$ and $B_t^{t, t'}$, where we have set $\delta = t' - t$ and introduced the martingales

$A_u^{t,t'} := N_{t+u}^{t'}(\ell) - N_t^{t'}(\ell)$, $u \leq \delta$ and $B_s^{t,t'} := N_s^{t'}(\ell) - N_s^t(\ell)$, $s \leq t$. Recall that $b^N(t', \ell)$ is of order 1 in $B_\epsilon^N(t')$. Since

$$A_u^{t,t'}(\ell) = \int_t^{t+u} \sum_k p_{t'-r}^N(k, \ell) dM^N(r, k),$$

a simple computation, using Proposition 5.3 and Lemma 6.3, shows that

$$\begin{aligned} \mathbb{E} \left[\langle A_{\cdot}^{t,t'}(\ell) \rangle_\delta^n \right]^{\frac{1}{n}} &\lesssim (2N)^{2\alpha} \int_t^{t'} \sum_k q_{r,t'}^N(k, \ell)^2 \mathbb{E} \left[\left(\frac{\xi^N(r, k)}{b^N(r, k)} \right)^{2n} \right]^{\frac{1}{n}} dr \\ &\lesssim (2N)^{2\alpha} \int_t^{t'} \frac{1}{\sqrt{t'-r} (2N)^{2\alpha}} dr \lesssim \sqrt{\delta}, \end{aligned}$$

uniformly over all $t < t' \in K$, all $\ell \in B_\epsilon^N(t')$ and all $N \geq 1$. Then, we set $D_u^{t,t'}(\ell) := [A_{\cdot}^{t,t'}(\ell)]_u - \langle A_{\cdot}^{t,t'}(\ell) \rangle_u$. Let J_k be the set of jump times of $\xi^N(\cdot, k)$. We have the almost sure bound

$$[D_{\cdot}^{t,t'}(\ell)]_\delta \lesssim \gamma_N^4 \sum_k \sum_{\tau \in (t,t'] \cap J_k} q_{\tau,t'}^N(k, \ell)^4 \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4,$$

uniformly over all the parameters. Thus, the same computation as in the proof of Lemma 5.4 ensures that

$$\mathbb{E} \left[[D_{\cdot}^{t,t'}(\ell)]_\delta^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim \gamma_N^4,$$

uniformly over all $t < t' \in K$, all $\ell \in B_\epsilon^N(t')$ and all $N \geq 1$. Thus, by (6.2), we deduce that

$$\mathbb{E} [|N_{t'}^{t'}(\ell) - N_t^{t'}(\ell)|^{2n}]^{\frac{1}{2n}} \lesssim |t' - t|^{\frac{1}{4}} + \frac{1}{(2N)^\alpha},$$

uniformly over the same set of parameters.

We turn our attention to $B^{t,t'}$. First, we have the identity

$$B_s^{t,t'}(\ell) = \int_0^s \sum_k (p_{t'-r}^N(k, \ell) - p_{t-r}^N(k, \ell)) dM^N(r, k), \quad \forall s \leq t,$$

so that

$$\mathbb{E} \left[\langle B_{\cdot}^{t,t'}(\ell) \rangle_\delta^n \right]^{\frac{1}{n}} \lesssim (2N)^{2\alpha} \int_0^t \sum_{k=1}^{2N-1} (q_{r,t'}^N(k, \ell) - q_{r,t}^N(k, \ell))^2 dr.$$

At this point, we argue differently according as k belongs to $B_{\epsilon/2}^N(r)$ or not. Using Lemma 6.4, we have

$$(2N)^{2\alpha} \int_0^t \sum_{k \notin B_{\epsilon/2}^N(r)} (q_{r,t'}^N(k, \ell) - q_{r,t}^N(k, \ell))^2 dr \lesssim N^{1+2\alpha} e^{-\delta N^{2\alpha}} \lesssim N^{-\alpha},$$

uniformly over all $\ell \in B_\epsilon^N(t')$, all $t < t' \in K$ and all $N \geq 1$. On the other hand, using Lemma 6.1, we get for all $\beta \in (0, 1/4)$

$$(2N)^{2\alpha} \int_0^t \sum_{k \in B_{\epsilon/2}^N(r)} (p_{r,t'}^N(k, \ell) - p_{r,t}^N(k, \ell))^2 dr \lesssim \int_0^t \frac{|t' - t|^{2\beta}}{(t-r)^{\frac{1}{2}+2\beta}} dr \lesssim (t' - t)^{2\beta},$$

uniformly over all $t < t' \in K$, all $\ell \in B_\epsilon^N(t')$ and all $N \geq 1$. Furthermore, we set $E_s^{t,t'} := [B_s^{t,t'}]_s - \langle B_s^{t,t'} \rangle_s$, $s \leq t$ and we have the almost sure bound

$$[E^{t,t'}(\ell)]_t \lesssim \gamma_N^4 \sum_{\tau \in (0,t]} \sum_k (q_{\tau,t'}^N(k, \ell) - q_{\tau,t}^N(k, \ell))^4 \left(\frac{\xi^N(\tau, k)}{b^N(\tau, k)} \right)^4,$$

uniformly over all $0 \leq t < t'$, all ℓ and all $N \geq 1$. This being given, we apply the same arguments as in the proof of Lemma 5.4 to get

$$\mathbb{E} \left[[E^{t,t'}(\ell)]_t^{\frac{n}{2}} \right]^{\frac{2}{n}} \lesssim \gamma_N^4,$$

uniformly over the same set of parameters. Using (6.2), we deduce that

$$\mathbb{E}[|N_t^{t'}(\ell) - N_t^t(\ell)|^{2n}]^{\frac{1}{2n}} \lesssim |t' - t|^\beta + \frac{1}{(2N)^\alpha},$$

uniformly over the same set of parameters, thus concluding the proof. \square

Proposition 5.6 *Fix $t_0 \in [0, T)$. The sequence ξ^N is tight in $\mathbb{D}([0, t_0], \mathcal{C}(\mathbf{R}))$, and any limit is continuous in time.*

Proof. One introduces a piecewise linear time-interpolation $\bar{\xi}^N$ of our process ξ^N , namely we set $t_N := \lfloor t(2N)^{4\alpha} \rfloor$ and

$$\bar{\xi}^N(t, \cdot) := (t_N + 1 - t(2N)^{4\alpha})\xi^N\left(\frac{t_N}{(2N)^{4\alpha}}, \cdot\right) + (t(2N)^{4\alpha} - t_N)\xi^N\left(\frac{t_N + 1}{(2N)^{4\alpha}}, \cdot\right).$$

Using Lemmas 5.4 and 5.5, it is simple to show that the space-time Hölder semi-norm of $\bar{\xi}^N$ on compact sets of $[0, T) \times \mathbf{R}$ has finite moments of any order, uniformly over all $N \geq 1$. Additionally, the proof of Lemma 4.7 in [BG97] carries through, and ensures that $\xi^N - \bar{\xi}^N$ converges to 0 uniformly over compact sets of $[0, T) \times \mathbf{R}$ in probability. All these arguments provide the required control on the space-time increments of ξ^N to ensure its tightness, following the calculation below Proposition 4.9 in [BG97]. \square

5.2 A delicate estimate

The goal of this section is to establish the following result, which is the analogue of Lemma 4.8 in [BG97]. We will use this bound to control error terms arising in the identification of the limit.

Proposition 5.7 *There exists $\kappa > 0$ such that for all $A > 0$, we have*

$$\mathbb{E} \left[|\mathbb{E}[\nabla^+ \xi^N(t, \ell) \nabla^- \xi^N(t, \ell) | \mathcal{F}_s]| \right] \lesssim \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{t-s}},$$

uniformly over all $\ell \in [N - A(2N)^{2\alpha}, N + A(2N)^{2\alpha}]$, all t in a compact set of $[N^{-\alpha}, T)$, all $s \in [0, t]$ and all $N \geq 1$.

To prove this proposition, we need to collect some preliminary results. Recall the decomposition (5.3). If we set

$$K_{t-r}^N(k, \ell) := \nabla^+ p_{t-r}^N(k, \ell) \nabla^- p_{t-r}^N(k, \ell),$$

(here the gradients act on the variable ℓ), then using the martingale property of $N^t(\ell)$ we obtain for all $s \leq t$

$$\begin{aligned} \mathbb{E} \left[\nabla^+ \xi^N(t, \ell) \nabla^- \xi^N(t, \ell) \mid \mathcal{F}_s \right] &= (\nabla^+ I^N(t, \ell) + \nabla^+ N_s^t(\ell)) (\nabla^- I^N(t, \ell) + \nabla^- N_s^t(\ell)) \\ &\quad + \mathbb{E} \left[\int_s^t \sum_{k=1}^{2N-1} K_{t-r}^N(k, \ell) d\langle M^N(\cdot, k) \rangle_r \mid \mathcal{F}_s \right]. \end{aligned}$$

Let

$$f_s^N(t, \ell) := \mathbb{E} \left[\left| \mathbb{E} \left[\nabla^+ \xi^N(t, \ell) \nabla^- \xi^N(t, \ell) \mid \mathcal{F}_s \right] \right| \right].$$

Fix $\epsilon > 0$. Using the expression of the bracket (5.2) of M^N , we get

$$f_s^N(t, \ell) \leq D_s^N(t, \ell) + \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} (2N)^{4\alpha} |K_{t-r}^N(k, \ell)| f_s^N(r, k) dr, \quad (5.13)$$

where $D_s^N(t, \ell) = D_s^{N,1}(t, \ell) + D_s^{N,2}(t, \ell) + D_s^{N,3}(t, \ell)$ with

$$\begin{aligned} D_s^{N,1}(t, \ell) &:= \mathbb{E} \left[|(\nabla^+ I^N(t, \ell) + \nabla^+ N_s^t(\ell)) (\nabla^- I^N(t, \ell) + \nabla^- N_s^t(\ell))| \right], \\ D_s^{N,2}(t, \ell) &:= \int_s^t \sum_{k \notin B_{\epsilon/2}^N(r)} (2N)^{4\alpha} |K_{t-r}^N(k, \ell)| f_s^N(r, k) dr, \\ D_s^{N,3}(t, \ell) &:= \lambda_N \mathbb{E} \left[\left| \mathbb{E} \left[\int_s^t \sum_{k=1}^{2N-1} K_{t-r}^N(k, \ell) (\xi^N(r, k) \Delta \xi^N(r, k) \right. \right. \right. \\ &\quad \left. \left. \left. + 2\xi^N(r, k)^2) dr \mid \mathcal{F}_s \right] \right| \right]. \end{aligned}$$

From now on, we fix a compact set $K \subset [0, T)$.

Lemma 5.8 *Fix $\epsilon > 0$. There exists $\kappa > 0$ such that*

$$D_s^N(t, \ell) \lesssim 1 \wedge \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{t-s}}, \quad (5.14)$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $N^{-\alpha} \leq s < t \in K$ and all $N \geq 1$.

Proof. Let us observe that we have the simple bound

$$f_s^N(r, k) \lesssim b^N(r, k)^2 \gamma_N^2, \quad (5.15)$$

uniformly over all the parameters. Recall also that $b^N(t, \ell)$ is of order 1 whenever $\ell \in B_\epsilon^N(t)$.

Let \bar{p}^N be the discrete heat kernel on the whole line \mathbf{Z} sped up by $2c_N$, see Appendix 6.2, and set $\bar{K}_t^N(k, \ell) = \nabla^+ \bar{p}_t^N(\ell - k) \nabla^- \bar{p}_t^N(\ell - k)$.

Bound of $D_s^{N,1}$. It suffices to bound the square of the L^2 -norms of $\nabla^\pm I^N(t, \ell)$ and $\nabla^\pm N_s^t(\ell)$. By Proposition 5.2, we deduce that $(\nabla^\pm I^N(t, \ell))^2 \lesssim N^{-3\alpha}$ uniformly over all $N^{-\alpha} \leq t \in K$, all $\ell \in B_\epsilon^N(t)$ and all $N \geq 1$.

We now treat $\nabla^+ N_s^t(\ell)$ (the proof is the same with ∇^-). Using again Lemmas 6.4 and 6.5, we have

$$\begin{aligned} \mathbb{E}[(\nabla^+ N_s^t(\ell))^2] &\lesssim \mathbb{E}\left[\sum_{k=1}^{2N-1} \int_0^s (\nabla^+ p_{t-r}^N(k, \ell))^2 d\langle M(\cdot, k) \rangle_r\right] \\ &\lesssim (2N)^{2\alpha} \sum_{k \in B_{\epsilon/2}^N(r)} \int_0^s (\nabla^+ \bar{p}_{t-r}^N(k, \ell))^2 dr + \mathcal{O}(N^{1+2\alpha} e^{-\delta N^{2\alpha}}), \end{aligned}$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $t \in K$ and all $N \geq 1$. Using Lemma 6.2, we easily deduce that the last expression is bounded by a term of order $1 \wedge 1/(\sqrt{t-s}(2N)^{4\alpha})$ as required.

Bound of $D^{N,2}$. Using the exponential decay of Lemma 6.4 and (5.15), we deduce that there exists $\delta > 0$ such that

$$\int_s^t \sum_{k \notin B_{\epsilon/2}^N(r)} (2N)^{4\alpha} |K_{t-r}^N(k, \ell)| f_s^N(r, k) dr \lesssim \int_s^t \sum_{k \notin B_{\epsilon/2}^N(r)} (2N)^{2\alpha} e^{-2\delta N^{2\alpha}} dr,$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $s \leq t \in K$ and all $N \geq 1$. This trivially yields a bound of order $N^{-3\alpha}$ as required.

Bound of $D^{N,3}$. By Lemmas 6.5 and 6.4, there exists $\delta > 0$ such that $D_s^{N,3}(t, \ell)$ can be rewritten as

$$\lambda_N \mathbb{E}\left[\left|\mathbb{E}\left[\int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) (\xi^N(r, k) \Delta \xi^N(r, k) + 2\xi^N(r, k)^2) dr \mid \mathcal{F}_s\right]\right|\right], \quad (5.16)$$

up to an error of order $N^{2\alpha+1} e^{-\delta N^{2\alpha}}$, uniformly over all $\ell \in B_\epsilon^N(t)$, all $t \in K$ and all $N \geq 1$. The error term satisfies the bound of the statement. We bound separately the two contributions arising in (5.16). First, using the almost sure bound $|\Delta \xi^N(r, k)| \lesssim \gamma_N \xi^N(r, k)$, we get

$$\begin{aligned} &\lambda_N \mathbb{E}\left[\left|\mathbb{E}\left[\int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) \xi^N(r, k) \Delta \xi^N(r, k) dr \mid \mathcal{F}_s\right]\right|\right] \\ &\lesssim (2N)^\alpha \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} |\bar{K}_{t-r}^N(k, \ell)| dr, \end{aligned}$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $t \in K$ and all $N \geq 1$. Using Lemma 6.2, this easily yields a bound of order $1/(2N)^{2\alpha+\kappa}$ with $\kappa > 0$, as required. Second, we have

$$\int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) \mathbb{E}[\xi^N(r, k)^2 \mid \mathcal{F}_s] dr$$

$$\begin{aligned}
&= \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) \mathbb{E} \left[\xi^N(r, k)^2 - \xi^N(t, \ell)^2 \mid \mathcal{F}_s \right] dr \\
&+ \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) dr \mathbb{E} \left[\xi^N(t, \ell)^2 \mid \mathcal{F}_s \right].
\end{aligned}$$

Using a simple integration by parts, we get

$$\int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) dr = \int_{t-s}^\infty \sum_{k \in \mathbb{Z}} \bar{K}_r^N(k, \ell) dr - \int_s^t \sum_{k \notin B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) dr.$$

The second term on the right can be bounded using Lemma 6.4: it has a negligible contribution. Using Lemma 6.2 on the first term, we easily deduce that

$$\lambda_N \mathbb{E} \left[\left| \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) dr \mathbb{E} \left[\xi^N(t, \ell)^2 \mid \mathcal{F}_s \right] \right| \right] \lesssim 1 \wedge \frac{1}{\sqrt{t-s}(2N)^{4\alpha}},$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $t \in K$ and all $N \geq 1$. On the other hand, for any given $\beta \in (0, 1/4)$, the Cauchy-Schwarz inequality together with Lemmas 5.4 and 5.5 yields

$$\begin{aligned}
\mathbb{E} \left[|\xi^N(r, k)^2 - \xi^N(t, \ell)^2| \right] &\lesssim \mathbb{E} \left[(\xi^N(r, k) + \xi^N(r, \ell))^2 \right]^{\frac{1}{2}} \mathbb{E} \left[(\xi^N(r, k) - \xi^N(r, \ell))^2 \right]^{\frac{1}{2}} \\
&+ \mathbb{E} \left[(\xi^N(r, \ell) + \xi^N(t, \ell))^2 \right]^{\frac{1}{2}} \mathbb{E} \left[(\xi^N(r, \ell) - \xi^N(t, \ell))^2 \right]^{\frac{1}{2}} \\
&\lesssim 1 \wedge \left(\left| \frac{\ell - k}{(2N)^{2\alpha}} \right|^{2\beta} + |t - r|^\beta + \frac{1}{(2N)^\alpha} \right),
\end{aligned}$$

uniformly over all $\ell \in B_\epsilon^N(t)$, all $k \in B_{\epsilon/2}^N(r)$, all $r \leq t \in K$ and all $N \geq 1$. Using Lemma 6.2, it is simple to deduce the existence of $\kappa \in (0, 1)$ such that

$$\lambda_N \mathbb{E} \left[\left| \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} \bar{K}_{t-r}^N(k, \ell) \mathbb{E} \left[\xi^N(r, k)^2 - \xi^N(t, \ell)^2 \mid \mathcal{F}_s \right] dr \right| \right] \lesssim \frac{1}{(2N)^{2\alpha+\kappa}},$$

uniformly over all $s < t \in K$, all $\ell \in B_\epsilon^N(t)$ and all $N \geq 1$. \square

We have all the elements at hand to prove the main result of this section.

Proof of Proposition 5.7. Iterating (5.13) and using Lemma 6.6 and the bound (5.15), we deduce that

$$f_s^N(t, \ell) \leq D_s^N(t, \ell) + \sum_{n \geq 1} H_s(t, \ell, n),$$

where for all $n \geq 1$, we set $t_{n+1} = t$, $k_{n+1} = \ell$ and

$$H_s(t, \ell, n) := \int_{s \leq t_1 \leq \dots \leq t_n \leq t} \sum_{k_i \in B_{\epsilon/2}^N(t_i)} D_s^N(t_1, k_1) \prod_{i=1}^n (2N)^{4\alpha} |K_{t_{i+1}-t_i}^N(k_i, k_{i+1})| dt_i.$$

By Lemma 5.8, we already know that $D_s^N(t, \ell)$ satisfies the bound of the statement of Proposition 5.7. To conclude the proof of the proposition, we only need to show that this is also the case for the sum over $n \geq 1$ of $H_s(t, \ell, n)$.

Fix $A > 0$. Let $n_0 = c \log N$, for an arbitrary $c > -3\alpha/\log \beta$, where $\beta < 1$ is taken from Lemma 6.6. Using Lemmas 5.8 and 6.6, we easily deduce that $H_s(t, \ell, n) \lesssim \beta^n$ uniformly over all $n \geq 1$, all $\ell \in \{N - A(2N)^{2\alpha}, N + A(2N)^{2\alpha}\}$ and all $N^{-\alpha} \leq s \leq t \in K$. Given the definition of n_0 , we deduce that

$$\sum_{n \geq n_0} I_s(t, \ell, n) \lesssim (2N)^{-3\alpha},$$

uniformly over the same set of parameters, as required.

Let us now treat $\sum_{n < n_0} H_s(t, \ell, n)$. We introduce

$$A_s(t, \ell, n) := \int_{s \leq t_1 \leq \dots \leq t_n \leq t} \sum_{k_i \in B_\epsilon^N(t_i)} D_s^N(t_1, k_1) \prod_{i=1}^n (2N)^{4\alpha} |K_{t_{i+1}-t_i}^N(k_i, k_{i+1})| dt_i.$$

By Lemma 5.8, we have

$$\begin{aligned} A_s(t, \ell, n) &\lesssim \int_{s \leq t_1 \leq \dots \leq t_n \leq t} \sum_{k_i \in B_\epsilon^N(t_i)} \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{t_1-s}} \\ &\quad \times \prod_{i=1}^n (2N)^{4\alpha} |K_{t_{i+1}-t_i}^N(k_i, k_{i+1})| dt_i. \end{aligned}$$

If we restrict the domain of integration to those t_1 such that $t_1 - s \geq (t-s)/(n+1)$, then a simple calculation based on Lemma 6.6 ensures that this restricted integral is bounded by a term of order

$$\frac{\sqrt{n+1} \beta^n}{(2N)^{2\alpha+\kappa} \sqrt{t-s}} \lesssim \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{t-s}},$$

for all $n \leq n_0$. On the other hand, when $t_1 - s < (t-s)/(n+1)$ there is at least one increment $t_{i+1} - t_i$ which is larger than $(t-s)/(n+1)$. By symmetry, let us assume that $i = 1$. By Lemma 6.5, we can replace $K_{t_2-t_1}^N(k_1, k_2)$ with $\bar{K}_{t_2-t_1}^N(k_1, k_2)$ up to a negligible term. By Lemma 6.2, we bound $|\bar{K}_{t_2-t_1}^N(k_1, k_2)|$ by a term of order $(2N)^{-4\alpha}(t_2 - t_1)^{-1}$ thus yielding the bound

$$\begin{aligned} &\int_s^{s+\frac{t-s}{n+1}} \sum_{k_1 \in B_\epsilon^N(t_1)} \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{t_1-s}} (2N)^{4\alpha} |K_{t_2-t_1}^N(k_1, k_2)| dt_1 \\ &\lesssim \int_s^{s+\frac{t-s}{n+1}} \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{t_1-s} (t_2 - t_1)} dt_1 + \mathcal{O}(N^{1+4\alpha} e^{-\delta N^{2\alpha}}) \\ &\lesssim \frac{\sqrt{\frac{t-s}{n+1}}}{(2N)^{2\alpha+\kappa} \frac{t-s}{n+1}} \lesssim \frac{1}{(2N)^{2\alpha+\kappa'} \sqrt{t-s}}, \end{aligned}$$

for all $\kappa' \in (0, \kappa)$ and all $n < n_0$. Using Lemma 6.6, we can bound the integral over t_2, \dots, t_n of the remaining terms by a term of order β^{n-1} . Consequently, we have proved that there exists $\kappa' > 0$ such that

$$A_s(t, \ell, n) \lesssim \frac{\beta^{n-1}}{(2N)^{2\alpha+\kappa'} \sqrt{t-s}}, \quad (5.17)$$

uniformly over all $\ell \in [N - A(2N)^{2\alpha}, N + A(2N)^{2\alpha}]$, all $t \in K$, all $s \in [0, t]$, all $n < n_0$ and all $N \geq 1$.

Finally, we set $B_s(t, \ell, n) := I_s(t, \ell, n) - A_s(t, \ell, n)$. As for the previous term, we can replace each occurrence of p^N by \bar{p}^N up to a negligible term, using Lemma 6.5. Among the parameters k_1, \dots, k_n involved in the definition of $B_s(t, \ell, n)$, at least one them, say k_{i_0} , belongs to $B_{\epsilon/2}^N(t_{i_0}) \setminus B_\epsilon^N(t_{i_0})$. Then, using the bound $|\bar{K}_t^N(k, \ell)| \leq \bar{p}_t^N(k, \ell)$ together with the semigroup property of the discrete heat kernel at the second line and the exponential decay of Lemma 6.4, we get

$$\begin{aligned} & \sum_{k_{i_0+1}, \dots, k_n} \prod_{i=i_0}^n |\bar{K}_{t_{i+1}-t_i}^N(k_i, k_{i+1})| \\ & \leq \sum_{k_{i_0+1}, \dots, k_n} \prod_{i=i_0}^n \bar{p}_{t_{i+1}-t_i}^N(k_i, k_{i+1}) = \bar{p}_{t-t_{i_0}}^N(k_{i_0}, \ell) \lesssim e^{-\delta N^{2\alpha}}, \end{aligned}$$

uniformly over all the parameters. Using Lemma 5.8, one easily gets

$$B_s(t, \ell, n) \lesssim (2N)^{n4\alpha} e^{-\delta(2N)^{2\alpha}},$$

uniformly over all $n \geq 1$, all $s < t \in K$ and all $\ell \in [N - A(2N)^{2\alpha}, N + A(2N)^{2\alpha}]$. Given the definition of n_0 , we deduce that the sum over all $n < n_0$ of the latter is negligible w.r.t. $(2N)^{-3\alpha}$, uniformly over the same set of parameters. This concludes the proof. \square

5.3 Identification of the limit

We use the notation $\langle f, g \rangle$ to denote the inner product of f and g in $L^2(\mathbf{R})$. Similarly, for all maps $f, g : [0, 2N] \rightarrow \mathbf{R}$, we set

$$\langle f, g \rangle_N := \frac{1}{(2N)^{2\alpha}} \sum_{k=1}^{2N-1} f\left(\frac{k-N}{(2N)^{2\alpha}}\right) g\left(\frac{k-N}{(2N)^{2\alpha}}\right).$$

To conclude the proof of Theorem 1.8, it suffices to show that any limit point ξ of a converging subsequence of ξ^N satisfies the following martingale problem (see Proposition 4.11 in [BG97]).

Definition 5.9 (Martingale problem) Let $(\xi(t, x), t \in [0, T], x \in \mathbf{R})$ be a continuous process satisfying the following two conditions. Let $t_0 \in [0, T]$. First, there exists $a > 0$ such that

$$\sup_{t \leq t_0} \sup_{x \in \mathbf{R}} e^{-a|x|} \mathbb{E}[\xi(0, x)^2] < \infty.$$

Second, for all $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$, the processes

$$\begin{aligned} M(t, \varphi) &:= \langle \xi(t), \varphi \rangle - \langle \xi(0), \varphi \rangle - \frac{1}{2} \int_0^t \langle \xi(s), \varphi'' \rangle ds, \\ L(t, \varphi) &:= M(t, \varphi)^2 - 4\sigma^2 \int_0^t \langle \xi(s)^2, \varphi^2 \rangle ds, \end{aligned}$$

are local martingales on $[0, t_0]$. Then, ξ is a solution of (1.5) on $[0, T]$.

The first condition is a simple consequence of Proposition 5.3. To prove that the second condition is satisfied, we introduce the discrete analogues of the above processes. For all $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$, the processes

$$\begin{aligned} M^N(t, \varphi) &= \langle \xi^N(t), \varphi \rangle_N - \langle \xi(0), \varphi \rangle_N - \frac{1}{2} \int_0^t \langle \xi(s), (2N)^{4\alpha} \Delta \varphi \rangle_N ds, \\ L^N(t, \varphi) &= M^N(t, \varphi)^2 - \frac{2\lambda_N}{(2N)^{2\alpha}} \int_0^t \langle \xi^N(s)^2, \varphi^2 \rangle_N ds + R_1^N(t, \varphi) + R_2^N(t, \varphi), \end{aligned}$$

are martingales, where

$$\begin{aligned} R_1^N(t, \varphi) &:= -\frac{\lambda_N}{(2N)^{2\alpha}} \int_0^t \langle \xi^N(s) \Delta \xi^N(s), \varphi^2 \rangle_N ds, \\ R_2^N(t, \varphi) &:= (2N)^{2\alpha} \int_0^t \langle \nabla^+ \xi^N(s) \nabla^- \xi^N(s), \varphi^2 \rangle_N ds. \end{aligned}$$

If we show that $R_1^N(t, \varphi)$ and $R_2^N(t, \varphi)$ vanish in probability when $N \rightarrow \infty$, then passing to the limit on a converging subsequence, we easily deduce that the martingale problem above is satisfied. Below, we will be working on $[N - A(2N)^{2\alpha}, N + A(2N)^{2\alpha}]$ where A is a large enough value such that $[-A, A]$ contains the support of φ . The moments of ξ^N on this interval are of order 1 thanks to Proposition 5.3. Since $|\Delta \xi^N| \lesssim \gamma_N \xi^N$, we have

$$\mathbb{E}[|R_1^N(t, \varphi)|] \lesssim \gamma_N \int_0^t \frac{1}{(2N)^{2\alpha}} \sum_k \varphi^2\left(\frac{k-N}{(2N)^{2\alpha}}\right) ds \lesssim \gamma_N,$$

so that $R_1^N(t, \varphi)$ converges to 0 in probability as $N \rightarrow \infty$. To control R_2^N , we write

$$\begin{aligned} \mathbb{E}[R_2^N(t, \varphi)^2] &\lesssim \int_0^t \int_0^s \sum_{k, k'} \varphi^2\left(\frac{k-N}{(2N)^{2\alpha}}\right) \varphi^2\left(\frac{k'-N}{(2N)^{2\alpha}}\right) \\ &\quad \times \left| \mathbb{E}[\nabla^+ \xi^N(s, k) \nabla^- \xi^N(s, k) \nabla^+ \xi^N(s', k') \nabla^- \xi^N(s', k')] \right| ds ds'. \end{aligned}$$

By the Cauchy-Schwarz inequality, for all $C > 0$ we have

$$\begin{aligned} &|\mathbb{E}[\mathbf{1}_{\{\xi^N(s', k') > C\}} \nabla^+ \xi^N(s, k) \nabla^- \xi^N(s, k) \nabla^+ \xi^N(s', k') \nabla^- \xi^N(s', k')]| \\ &\lesssim \gamma_N^4 \mathbb{P}(\xi^N(s', k') > C)^{\frac{1}{2}} \mathbb{E}[\xi^N(s, k)^4 \xi^N(s', k')^4]^{\frac{1}{2}} \lesssim \gamma_N^4 \frac{1}{\sqrt{C}}, \end{aligned}$$

uniformly over all $k, k' \in [N - A(2N)^{2\alpha}, N + A(2N)^{2\alpha}]$ and all $s, s' \in [0, t]$. On the other hand, by Proposition 5.7 there exists $\kappa > 0$ such that for all $C > 0$ we have

$$\begin{aligned} &|\mathbb{E}[\mathbf{1}_{\{\xi^N(s', k') \leq C\}} \nabla^+ \xi^N(s, k) \nabla^- \xi^N(s, k) \nabla^+ \xi^N(s', k') \nabla^- \xi^N(s', k')]| \\ &\lesssim \gamma_N^2 \mathbb{E}[\mathbf{1}_{\{\xi^N(s', k') \leq C\}} \xi^N(s', k')^2] \left| \mathbb{E}[\nabla^+ \xi^N(s, k) \nabla^- \xi^N(s, k) | \mathcal{F}_{s'}] \right| \\ &\lesssim C^2 \gamma_N^2 \frac{1}{(2N)^{2\alpha+\kappa} \sqrt{s-s'}}. \end{aligned}$$

uniformly over all k, k' , all $s \in [0, t]$, all $s' \in [N^{-\alpha}, s]$ and all $N \geq 1$. This being given, we easily deduce that

$$\mathbb{E}[R_2^N(t, \varphi)^2] \lesssim \frac{t}{N^\alpha} + \frac{t^2}{\sqrt{C}} + \frac{C^2 t^{3/2}}{(2N)^\kappa},$$

uniformly over all $C > 0$ and all $N \geq 1$. Taking the limit as $N \rightarrow \infty$ and then as $C \rightarrow \infty$, we deduce that $R_2^N(t, \varphi)$ converges to 0 in probability, thus concluding the proof.

6 Appendix

6.1 Martingale inequalities

Let $X(t), t \geq 0$ be a càdlàg, mean zero, square-integrable martingale. Let $\langle X \rangle_t, t \geq 0$ denote the bracket of X , that is, the unique predictable process such that $X^2 - \langle X \rangle$ is a martingale. Let $[X]_t$ denote its quadratic variation: in the case where the martingale is of finite variation, we have

$$[X]_t = \sum_{\tau \in (0, t]} (X_\tau - X_{\tau-})^2.$$

The Burkholder-Davis-Gundy inequality ensures that for every $p \geq 1$, there exists $c(p) > 0$ such that

$$\mathbb{E}[|X_t|^p]^{\frac{1}{p}} \leq c(p) \mathbb{E}\left[[X]_t^{\frac{p}{2}}\right]^{\frac{1}{p}}. \quad (6.1)$$

It happens that the process $D_t = [X]_t - \langle X \rangle_t$ is also a martingale. Thus, using twice the Burkholder-Davis-Gundy inequality, one gets that for every $p \geq 2$ there exists $c'(p) > 0$ such that

$$\mathbb{E}[|X_t|^p]^{\frac{1}{p}} \leq c'(p) \left(\mathbb{E}\left[\langle X \rangle_t^{\frac{p}{2}}\right]^{\frac{1}{p}} + \mathbb{E}\left[[D]_t^{\frac{p}{4}}\right]^{\frac{1}{p}} \right). \quad (6.2)$$

We will also rely on the following inequality

$$\mathbb{E}\left[\sup_{s \leq t} |X_s|^p\right]^{\frac{1}{p}} \leq c''(p) \left(\mathbb{E}\left[\langle X \rangle_t^{\frac{p}{2}}\right]^{\frac{1}{p}} + \mathbb{E}\left[\sup_{s \leq t} |X_s - X_{s-}|^p\right]^{\frac{1}{p}} \right), \quad (6.3)$$

which can be found in [LLP80] for instance.

6.2 Discrete heat kernel estimates

We introduce the fundamental solution $p_t^N(k, \ell)$ of the discrete heat equation

$$\begin{cases} \partial_t p_t^N(k, \ell) = c_N \Delta p_t^N(k, \ell), \\ p_0^N(k, \ell) = \delta_k(\ell), \\ p_t^N(k, 0) = p_t^N(k, 2N) = 0, \end{cases} \quad (6.4)$$

for all $k, \ell \in \{1, \dots, 2N-1\}$, as well as its analogue $\bar{p}_t^N(\ell)$ on \mathbf{Z} :

$$\begin{cases} \partial_t \bar{p}_t^N(\ell) = c_N \Delta \bar{p}_t^N(\ell), \\ \bar{p}_0^N(\ell) = \delta_0(\ell), \end{cases} \quad (6.5)$$

for all $\ell \in \mathbf{Z}$. The latter is more tractable than the former since it is translation invariant. Using a coupling between a simple random walk on \mathbf{Z} and a simple random walk killed at 0 and $2N$, we get the elementary bound $p_t^N(k, \ell) \leq \bar{p}_t^N(\ell - k)$ for all $k, \ell \in \{1, \dots, 2N - 1\}$ and all $t \geq 0$. The following estimates are classical, see for instance Lemma A.1 in [DT16] or Lemma 26 in [EL15].

Lemma 6.1 *For all $\beta \in [0, 1]$, we have*

$$\begin{aligned} p_t^N(k, \ell) &\lesssim 1 \wedge \frac{1}{\sqrt{tc_N}}, \\ |p_t^N(k, \ell) - p_t^N(k, \ell')| &\lesssim 1 \wedge \frac{1}{\sqrt{tc_N}} \left| \frac{\ell - \ell'}{\sqrt{c_N}} \right|^\beta, \\ |p_t^N(k, \ell) - p_{t'}^N(k, \ell)| &\lesssim 1 \wedge \frac{1}{\sqrt{tc_N}} \left| \frac{t - t'}{t} \right|^\beta, \end{aligned}$$

uniformly over all $0 \leq t < t'$, all $k, \ell, \ell' \in \{1, \dots, 2N - 1\}$ and all $N \geq 1$. The same bounds hold for \bar{p}^N .

Let us also state the following simple bounds.

Lemma 6.2 *We have $\sum_k \bar{p}_t^N(k)|k| \lesssim \sqrt{c_N t}$ as well as*

$$\sum_{k \in \mathbf{Z}} |\nabla \bar{p}_t^N(k)| \leq 2\bar{p}_t^N(0), \quad \sum_{k \in \mathbf{Z}} |\nabla \bar{p}_t^N(k)||k| \lesssim 1, \quad |\nabla \bar{p}_t^N(\ell)| \lesssim 1 \wedge \frac{1}{tc_N},$$

uniformly over all $\ell \in \mathbf{Z}$, all $t \geq 0$ and all $N \geq 1$.

Proof. Notice that $\sum_k \bar{p}_t^N(k)|k|$ is smaller than the square root of the variance of a simple random walk on \mathbf{Z} at time $2c_N t$. This easily yields the first bound. We turn to the bounds involving the gradient of \bar{p}^N . First, $\nabla \bar{p}_t^N(k)$ is positive if $k < 0$, and negative otherwise. Then, we have the simple identity

$$\sum_{k < 0} \nabla \bar{p}_t^N(k) = - \sum_{k \geq 0} \nabla \bar{p}_t^N(k) = \bar{p}_t^N(0),$$

which yields the first bound. Regarding the second bound, a simple integration by parts yields

$$- \sum_{k < 0} \nabla \bar{p}_t^N(k)k = \sum_{k \leq 0} \bar{p}_t^N(k), \quad - \sum_{k \geq 0} \nabla \bar{p}_t^N(k)k = \sum_{k > 0} \bar{p}_t^N(k),$$

so that we get $\sum_{k \in \mathbf{Z}} |\nabla \bar{p}_t^N(k)||k| = 1$. To get the third bound, it suffices to use the Fourier decomposition of \bar{p}^N . \square

From now on, we work in the setting of Section 5. Recall the definition of q^N from (5.7).

Lemma 6.3 *Uniformly over all $0 \leq s < t$, all $\ell \in \{1, \dots, 2N - 1\}$ and all $N \geq 1$, we have*

$$\sum_{k=1}^{2N-1} q_{s,t}^N(k, \ell) \lesssim b^N(t, \ell), \quad q_{s,t}^N(k, \ell) \lesssim b^N(t, \ell) \left(1 \wedge \frac{1}{\sqrt{t-s} (2N)^{2\alpha}} \right). \quad (6.6)$$

Proof. By symmetry, it is sufficient to prove the lemma under the further assumption $\ell \leq N$. If $b^N(s, k) \leq 3$, then we have $b^N(s, k)/b^N(t, \ell) \leq 3$ while if $b^N(s, k) > 3$, then a simple calculation ensures that

$$\frac{b^N(s, k)}{b^N(t, \ell)} \lesssim \begin{cases} e^{-\lambda_N(t-s)+\gamma_N(\ell-k)} & \text{if } k \leq N, \\ e^{-\lambda_N(t-s)+\gamma_N(k-\ell)} & \text{if } k \geq N, \end{cases}$$

uniformly over all $s < t$, all $k \in \{1, \dots, 2N-1\}$, all $\ell \in \{1, \dots, N\}$ and all $N \geq 1$. Therefore, it suffices to show that

$$\sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell) \lesssim 1, \quad p_{t-s}^N(k, \ell) \lesssim 1 \wedge \frac{1}{\sqrt{t-s}(2N)^{2\alpha}}, \quad (6.7)$$

as well as

$$\begin{aligned} \sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell) e^{-\lambda_N(t-s)+\gamma_N|\ell-k|} &\lesssim 1, \\ p_{t-s}^N(k, \ell) e^{-\lambda_N(t-s)+\gamma_N|\ell-k|} &\lesssim 1 \wedge \frac{1}{\sqrt{t-s}(2N)^{2\alpha}}. \end{aligned} \quad (6.8)$$

Regarding (6.7), the first bound is immediate since $p_t^N(\cdot, \ell)$ is a sub-probability measure, while the second bound is proved in Lemma 6.1. We turn to (6.8). Since $k \mapsto e^{ak}$ is an eigenvector of the discrete Laplacian on \mathbf{Z} with eigenvalue $2(\cosh a - 1)$, we deduce that

$$\begin{aligned} \sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell) e^{\gamma_N|\ell-k|} &\leq \sum_{k \in \mathbf{Z}} \bar{p}_{t-s}^N(\ell - k) e^{\gamma_N|\ell-k|} \\ &\leq 2e^{2c_N(t-s)(\cosh \gamma_N - 1)} = 2e^{\lambda_N(t-s)}, \end{aligned}$$

thus yielding the first bound. To get the second bound, it suffices to use the Fourier decomposition of $\bar{p}_{t-s}^N(\cdot) e^{\gamma_N \cdot}$. \square

We also recall a simple bound on the heat kernel, in the flavour of large deviations techniques. For all $a > 0$, $t > 0$ and $N \geq 1$, we have

$$\sum_{k \geq a} \bar{p}_t^N(k) \leq e^{2tc_N g(\frac{a}{2c_N t})}, \quad (6.9)$$

where $g(x) = \cosh(\operatorname{argsh} x) - x \operatorname{argsh} x - 1$ for all $x \in \mathbf{R}$. By studying the function $g(x)/x$, one easily deduces that the term on the r.h.s. is increasing with t .

We let $\bar{q}_{s,t}^N(k, \ell) := \bar{p}_{t-s}^N(k, \ell) b^N(s, k)$.

Lemma 6.4 *Fix a compact set $K \subset [0, T)$ and $\epsilon > 0$. There exists $\delta > 0$ such that*

$$q_{s,t}^N(k, \ell) \leq \bar{q}_{s,t}^N(k, \ell) \lesssim e^{-\delta N^{2\alpha}},$$

uniformly over all $k \notin B_{\epsilon/2}^N(s)$, all $\ell \in B_\epsilon^N(t)$ and all $0 \leq s \leq t \in K$.

Proof. Let us consider the case where $b^N(s, k) \geq 3$; by symmetry we can assume that $k \in \{1, \dots, N\}$. Then, we apply (6.9) to get

$$\bar{p}_{t-s}^N(\ell - k)e^{\lambda_N s - \gamma_N k} \leq e^{2(t-s)c_N g(\frac{\ell-k}{2c_N(t-s)}) + \lambda_N s - \gamma_N k}. \quad (6.10)$$

We argue differently according to the value of α . If $4\alpha \leq 1$, then $(\ell - k)/c_N$ is bounded away from 0 uniformly over all $N \geq 1$, all $k \notin B_{\epsilon/2}(s)$ and all $\ell \in B_\epsilon(t)$. Using the concavity of g , we deduce that there exists $d > 0$ such that the logarithm of the r.h.s. of (6.10) is bounded by

$$-d(\ell - k) + \lambda_N s - \gamma_N k \lesssim -d\epsilon \frac{N}{2},$$

thus concluding the proof in that case.

We now treat the case $4\alpha > 1$. Let $\eta > 0$. First, we assume that $s \in [0, t - \eta]$. For any $c > 1/4!$, we have $g(x) \leq -x^2/2 + cx^4$ for all x in a neighbourhood of the origin. Then, for N large enough we bound the logarithm of the r.h.s. of (6.10) by

$$f(s) = -\frac{1}{2} \frac{(\ell - k)^2}{2c_N(t-s)} + c \frac{(\ell - k)^4}{(2c_N(t-s))^3} + \lambda_N s - \gamma_N k.$$

A tedious but simple calculation shows the following. There exists $\delta' > 0$, only depending on ϵ , such that $\sup_{s \in [0, t-\eta]} f(s) \leq -\delta' N^{2\alpha}$ for all N large enough. This ensures the bound of the statement in the case where $s \in [0, t - \eta]$.

Using the monotonicity in t of (6.9), we easily deduce that for all $s \in [t - \eta, t]$, we have

$$\bar{p}_{t-s}^N(\ell - k)e^{\lambda_N s - \gamma_N k} \leq e^{f(t-\eta) + \lambda_N \eta}.$$

Recall that λ_N is of order $N^{2\alpha}$. Choosing $\eta < \delta'$ small enough and applying the bound obtained above, we deduce that the statement of the lemma holds true.

The case where $b^N(s, k)$ is smaller than 3 is simpler, one can adapt the above arguments to get the required bound. \square

Proof of (5.6). The quantity $1 - \sum_{k=1}^{2N-1} p_{t-s}^N(k, \ell)$ is equal to the probability that a simple random walk, sped up by $2c_N$ and started from ℓ , has hit 0 or $2N$ by time $t - s$. By the reflexion principle, this is smaller than twice

$$\sum_{k \geq \ell} \bar{p}_{t-s}^N(k) + \sum_{k \geq 2N-\ell} \bar{p}_{t-s}^N(k).$$

We restrict ourselves to bounding the first term, since one can proceed similarly for the second term. Using (6.9), we deduce that it suffices to bound $\exp(2(t-s)c_N g(\ell/(2(t-s)c_N)) + \lambda_N s)$. This is equal to the l.h.s. of (6.10) when $k = 0$, so that the required bound follows from the arguments presented in the last proof. \square

Finally, we rely on the following representation of p^N :

$$p_t^N(k, \ell) = \sum_{j \in \mathbb{Z}} \bar{p}_t^N(k + j4N - \ell) - \bar{p}_t^N(-k + j4N - \ell).$$

The next lemma shows that $p_t^N(k, \ell)$ can be replaced by $\bar{p}_t^N(\ell - k)$ up to some negligible term, whenever ℓ is in the ϵ -bulk at time t .

Lemma 6.5 Fix $\epsilon > 0$ and a compact set $K \subset [0, T)$. There exists $\delta > 0$ such that uniformly over all $s \leq t \in K$, all $k \in \{1, \dots, 2N - 1\}$, all $\ell \in B_\epsilon^N(t)$ and all $N \geq 1$, we have

$$|p_{t-s}^N(k, \ell) - \bar{p}_{t-s}^N(k, \ell)|b^N(s, k) \lesssim e^{-\delta N^{2\alpha}}.$$

Proof. We only consider the case where $b^N(s, k) > 3$ since the other case is simpler. Observe that there exists $C > 0$ such that $\log b^N(s, k) \leq CN^{2\alpha}$ for all $s \in K$ and all $k \in \{1, \dots, 2N - 1\}$. Arguing differently according to the relative values of 4α and 1, and using the bound (6.9), we deduce that there exists $j_0 \geq 1$ such that

$$\sum_{j \in \mathbf{Z}: |j| \geq j_0} \bar{p}_{t-s}^N(k + j4N - \ell)b^N(s, k) + \bar{p}_{t-s}^N(-k + j4N - \ell)b^N(s, k) \lesssim e^{-\delta N^{2\alpha}}, \quad (6.11)$$

uniformly over all $s \leq t \in K$, all $k \in \{1, \dots, 2N - 1\}$ and all $\ell \in B_\epsilon(t)$. On the other hand, the arguments in the proof of Lemma 6.4 yield that

$$\begin{aligned} \sum_{j \in \mathbf{Z}: |j| < j_0} \bar{p}_{t-s}^N(-k + j4N - \ell)b^N(s, k) &\lesssim e^{-\delta N^{2\alpha}}, \\ \sum_{j \in \mathbf{Z}: 0 < |j| < j_0} \bar{p}_{t-s}^N(k + j4N - \ell)b^N(s, k) &\lesssim e^{-\delta N^{2\alpha}}, \end{aligned}$$

uniformly over the same set of parameters, thus concluding the proof. \square

Lemma 6.6 Fix $\epsilon > 0$. There exist $\beta \in (0, 1)$ such that

$$\int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} |K_{t-r}^N(k, \ell)|(2N)^{4\alpha} dr < \beta,$$

uniformly over all $s \leq t \in K$, all $\ell \in B_\epsilon^N(t)$ and all N large enough.

Proof. Lemma 6.5 ensures that

$$\begin{aligned} \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} |K_{t-r}^N(k, \ell)|(2N)^{4\alpha} dr &= \int_s^t \sum_{k \in B_{\epsilon/2}^N(r)} |\bar{K}_{t-r}^N(k, \ell)|(2N)^{4\alpha} dr \\ &\quad + \mathcal{O}(N^{1+4\alpha} e^{-\delta N^{2\alpha}}), \end{aligned}$$

uniformly over all $\ell \in B_\epsilon(t)$, all $t \in K$ and all $N \geq 1$. Lemma A.3 in [BG97] ensures that the first term on the r.h.s. is smaller than some $\beta' \in (0, 1)$. Since the second term vanishes as $N \rightarrow \infty$, the bound of the statement follows. \square

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